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Charge multiplets and masses for E_{11}

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Abstract: The particle, string and membrane charge multiplets are derived in detail from the decomposition of the l_1 (charge) representation of E_{11} in three, four, five, six, seven and eight spacetime dimensions. A tension formula relating weights of the l_1 (charge) representation of E_{11} to the fundamental objects of M-theory and string theory is presented. The reliability of the formula is tested by reproducing the tensions of the content of the charge multiplets. The formula reproduces the masses for the pp-wave, M2, M5 and the KK-monopole from the low level content of the l_1 representation of E_{11}. Furthermore the tensions of all the Dp-branes of IIA and IIB theories are found in the relevant decomposition of the l_1 representation, with the string coupling constant and α' appearing with the expected powers. The formula leads to a classification of all the exotic, KK-brane charges of M-theory.

Keywords: Supergravity Models, Global Symmetries, M-Theory, String Duality.

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Contents

Introduction. The Kac-Moody algebra E_{11} has been conjectured to be the algebra describing the symmetries of M-theory [1]. The arguments used to make this conjecture were based upon previously unnoticed properties of D=11 supergravity, leading to its formulation as a nonlinear realisation which included the Borel generators of E_7. These arguments lead to an E_{11} algebra in eleven dimensions encoding the symmetries of M-theory. Generalised Kac-Moody algebras, such as E_{11}, are not well understood and analysis of their content is hampered by the lack of a simple way of applying the Serre relations to the putative generators of the algebra. However E_{11} belongs to a class of algebras known as Lorentzian Kac-Moody algebras which have the property that the deletion of one node of the defining Dynkin diagram leaves behind a set of Dynkin diagrams of finite dimensional groups [2]. Progress has been made by decomposing the algebra into representations of finite dimensional sub-algebras which are graded by a level and analysing the content. Analysing the low-level content of the adjoint representation of E_{11} in this way one can reproduce the dynamics of the bosonic sector of D=11 supergravity. In addition one finds all the bosonic fields of the D=11 supergravity theory at the lowest levels of the decomposition as well as the dual gravity field leading to a dualised gravity theory.

However in the original non-linear realisation of the E_{11} symmetry the translation generator had to be added in by hand. Space-time did not emerge from the adjoint representation of E_{11}. It was proposed that one could more naturally include spacetime by enlarging the algebra to include its first fundamental representation, or the l_1 representation, as well as the adjoint representation of E_{11} [3]. The success of this approach is that the translation generator is associated to the highest weight of the l_1 representation and appears at the lowest level in the l_1 representation. Furthermore one finds that the l_1 representation contains generators having the correct index structure to be interpreted as the central charges of the supersymmetry algebra, even though the arguments used to conjecture an E_{11} symmetry considered only the bosonic fields of supergravity.

The associations between the l_1 representation and the adjoint representation of E_{11} as well as other very-extended algebras have been studied in detail in [5]. It has been proposed that the full set of brane charges of M-theory are contained in the l_1 representation of E_{11} [3] and, in fact, for every generator of the adjoint representation of E_{11} one can associate a half-BPS brane solution [4] and in the l_1 representation one can find a corresponding generator associated with the conserved charge on the brane. Following the proposition that the l_1 representation contains the brane charges of M-theory, the brane charge multiplets in three dimensions which are predicted by U-duality [6 − 10] have been found inside the l_1 representation [11], and the particle multiplet has been identified in all dimensions from three to eight in [12]. To find the three dimensional charge multiplet the dimensional reduction on an 8-torus was carried out as a decomposition of the l_1 representation into representations of an $A_2 \otimes E_8$ sub-algebra. In this paper we further the arguments in favour of the relevance of the l_1 representation by deriving all the possible charge multiplets from its algebra in three to eight spacetime dimensions.

If the conjecture is true that string theory, and M-theory, do carry a Kac-Moody algebra one may hope to better understand the algebras by making connection with areas of string theory that are well understood. One direction forward is by the introduction of group theoretical constructions that can be argued to have a clear interpretation in terms of the string theory. The most fundamental properties in a physical theory are those that give rise to the simplest dimensionful quantities. Space and time are already present inside the algebraic construction, via the local Lorentzian symmetry algebra, but a natural interpretation for mass is missing. In the case of string theory a non-perturbative concept amenable to study is the tension of BPS p-branes, or the mass per unit volume. Indeed previously [6 − 10] a group theoretical tension formula has been constructed as an empirical tool to discover the charge multiplets of M-theory and string theory by application of U-duality symmetries. It is our principle aim in this paper to derive this tension formula in the context of E_{11} and to check our formula by duplicating the tensions found in the U-duality charge multiplets.

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A by-product of the tension formula that will be derived is the observation that most of the l_1 representation is associated to exotic charges carried by KK-branes, which are higher-dimensional generalisations of the KK-monopole and provide a higher dimensional origin to certain brane charges and KK-modes in lower dimensions (see, for example, [21, 22]). Some of these KK-brane charges have been observed as a consequence of U-duality in [6 − 10]. The l_1 representation of E_{11} includes all the exotic KK-brane charges expected by U-duality transformations, and these are organised into finite sets within the l_1 representation. The interpretation of KK-brane charges offers a new way to decompose the E_{11} algebra, and conversely the l_1 representation also offers a classification of all the KK-brane charges expected in M-theory. We will present the full set of the simplest class of KK-brane charges expected in M-theory, as well as in the IIA and IIB string theories.

The discussion in the paper will be split into two parts. First in sections 1 and 2 we will give explicit decompositions of the l_1 algebra. In the first section we will give the decomposition of the l_1 representation of E_{11} relevant to the eleven dimensional theory as well as the decomposition to the two ten-dimensional theories. We present this original decomposition using an explicit basis for the root lattice vector space that has not been used in the E_{11} literature previously that will greatly simplify our observations later in this paper. In section 2, we study the l_1 representation and identify how the charge multiplets are organised inside the representation, we then derive explicitly the charge multiplets of M-theory in three to eight dimensions. The abstract decomposition of the l_1 representation into representations of $A_{D-1} \otimes E_{11-D}$ for D spacetime dimensions is given in section 2.1. In section 2.2 we give the criteria for finding rank p charges in the A_{D-1} algebra and find the corresponding representations in the E_{11-D} algebra. In the remainder of section 3 we reproduce the particle, string and membrane charge multiplets in various dimensions and give the tensions of the exotic charges according to the formula that will be presented later in this paper. The tensions for the particle and string multiplets confirm the findings in the literature but the membrane multiplet charges have not been presented explicitly before. In the second part of the paper, commencing with section 3, we will present a tension formula and use it to derive the tensions associated to the fundamental objects of M-theory and string theory in section 3.3 from the roots of the l_1 representation, based on the computations in the section 1. Expressions for the tension associated to any root in the l_1 representation are given in section 3.3 and the simplest KK-brane charges are listed in their entirety.

1. The l_1 representation of E_{11}

E_{11} is described completely by its Dynkin diagram which is found by attaching three additional roots to the longest leg of the E_8 diagram, each extra simple root having the same length as any root of E_8.

By deleting the exceptional node, labelled eleven, one finds representations of the remaining A_{10} sub-algebra, graded by the level, which is the number of times the deleted simple root must be added to the A_{10} root to make it into an E_{11} root. Decomposing the algebra in this way one finds the adjoint representation of E_{11} and at the first few

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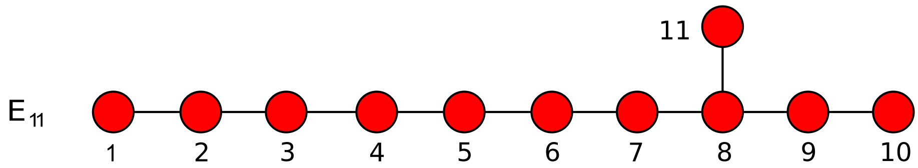

Figure 1: The Dynkin diagram of E_{11}.

levels one finds the gravitational field, a three-form field, a six-form field and the dual to gravity field. With the exception of the dual to gravity these fields are well-known bosonic fields from eleven dimensional supergravity. Dimensionally reducing the algebra to D dimensions corresponds to deleting different nodes of the Dynkin diagram leaving representations of A_{D-1} sub-algebras. For example in the reduction to ten dimensions there is choice of which nodes to delete and this gives rise to one of the most beautiful aspects of the construction, namely that the two choices correspond to the choice of IIA or IIB theories in ten dimensions. More explicitly one deletes nodes of the E_{11} Dynkin diagram so as to leave behind an A_9 Dynkin diagram (a line of nine connected nodes). One can do this in two ways, by deleting nodes 11 and 10 or by deleting node 9, which yields the bosonic fields of the IIA and IIB theories respectively in an elegant way.

The representations of E_{11} other than the adjoint are also interesting and of direct relevance to theoretical physics. The l_1, or charge, representation of E_{11} is believed to contain all the brane charges of M-theory in the E_{11} weight lattice [3]. The decomposition of this algebra to different spacetime dimensions also corresponds to the deletion of different nodes of its associated Dynkin diagram. Of interest to physicists are the decompositions giving representations of SL(11) and SL(10) sub-groups, which give generators in the algebra conjectured to be the brane charges of M-theory and the two ten-dimensional supergravity theories respectively. In section 1.1 we will give the decomposition relevant to M-theory and in sections 1.2 and 1.3 we make the decompositions connected to the IIA and IIB supergravity theories. The presentation in this section will differ cosmetically from much of the literature in that it will make use of a vector space basis denoted herein by $\{e_i\}$ vectors instead of the more usual simple root basis, α_i.

We recall that the l_1 representation of E_{11} takes the first fundamental weight of E_{11}, l_1, associated to the translation generator, and treats it as the highest weight of a representation in the E_{11} weight lattice [3]. Alternatively one can obtain the l_1 representation of E_{11} by extending the E_{11} Dynkin diagram with a node attached by a single line to the longest leg of the E_{11} diagram, giving the Dynkin diagram of E_{12} shown below, and one then restricts to just those roots with the coefficient of the extra root, α_*, set to one. In other words one decomposes E_{12} by the deletion of the node α_* and the l_1 representation of E_{11} is found at level one with highest weight l_1, the first fundamental weight of E_{11}.

A general root appearing in the E_{12} root lattice with $m_* = 1$ has the form:

$$\beta = \alpha_* + \sum_{i=1}^{11} m_i \alpha_i \tag{1.1}$$

Before beginning the decompositions of our algebra a few comments about the root lattices

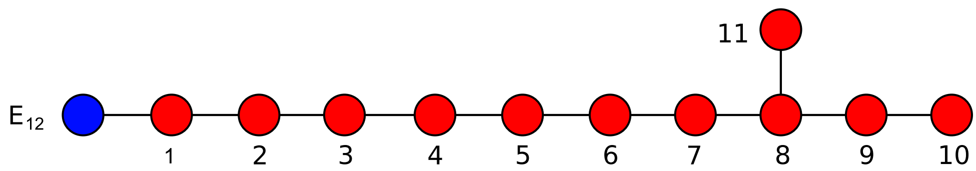

Figure 2: The Dynkin diagram of E_{12}.

of generalised Kac-Moody algebras are in order. Whether or not the generic root β appears in the root lattice depends upon the application of the Serre relations which, in terms of the Chevalley generators, are,

$$[E_i \ldots [E_i, E_j] \ldots] = 0 \qquad [F_i \ldots [F_i, F_j] \ldots] = 0 \tag{1.2}$$

Here there are $(1 - A_{ij})$ E_i generators (F_i generators in the second relation) where A_{ij} is the Cartan matrix associated to the Dynkin diagram of the algebra. One consequence of the Serre relations is that $\beta^2 \leq 2$ for roots in the l_1 lattice. Another consequence is that any root appearing in the lattice must have connected support on the Dynkin diagram. These and another readily defined condition, on the index structure of generators appearing in the algebra to be discussed later, are sufficient to capture much of the the Serre relations, and are far easier to apply computationally. The application of the Serre relations in a simple manner is an open problem in mathematics and later on when we give explicit roots in the lattice at high levels these are the criteria that will have been applied.

1.1 The eleven-dimensional theory

Our aim here is to decompose the l_1 representation into representations of a preferred A_{10} sub-algebra, giving representations of SL(11). This decomposition has been made previously [5] but here we will present the results using a vector space basis $\{e_*, e_1, \ldots e_{11}\}$ for the root lattice that will simplify later calculations in the paper but has not been much used in the literature previously. The preferred A_{10} algebra is given by the Dynkin diagram of E_{12} shown above when the nodes indicated by a $*$ and 11 are deleted and is called the gravity line in this decomposition. The generators will be SL(11) tensors graded by a level and are believed to be the brane charges of M-theory.

We first decompose the roots into components in the E_{11} lattice and a vector orthogonal to it, which we shall call y, by writing

$$\alpha_* = y - l_1 \tag{1.3}$$

Where l_1 is the first fundamental weight of E_{11}, we recall that fundamental weights are dual to the corresponding simple roots of an algebra:

$$< \lambda_i, \alpha_j >= \delta_{ij} \tag{1.4}$$

Where λ_i denotes the i'th fundamental weight and α_i are the simple roots indicated by the Dynkin diagram of the algebra. In this case, $\alpha_*^2 = 2$ and $l_1^2 = \frac{1}{2}$, so that $y^2 = \frac{3}{2}$. It will

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be useful later to consider an explicit vector space basis for our root system. We introduce the basis $\{e_*, e_1, \dots e_{11}\}$ endowed with the Lorentzian inner product:

$$< a, b >= \sum_{i=*}^{11} a_i b_i - \frac{1}{9} \sum_{i=*}^{11} a_i \sum_{j=*}^{11} b_j \tag{1.5}$$

Where $a = \sum_{i=*}^{11} a_i e_i$ and similarly for b. We represent the simple roots of E_{12} in this basis with,

$$\begin{aligned} \alpha_i &= e_i - e_{i+1} \qquad i = *, 1, 2 \dots 10 \\ \alpha_{11} &= e_9 + e_{10} + e_{11} \end{aligned} \tag{1.6}$$

So that the inner products between the simple roots encoded in the Cartan matrix are reproduced, and all roots have length-squared normalised to two. In this notation the vector y is,

$$y = e_* - \frac{1}{2}(e_1 + \cdots + e_{11}) \tag{1.7}$$

A generic root in the l_1 representation therefore takes the form:

$$\beta = y - l_1 + \sum_{i=1}^{11} m_i \alpha_i \tag{1.8}$$

And defines the weight vectors in the E_{11} lattice descended from the highest weight l_1:

$$\Lambda = l_1 - \sum_{i=1}^{11} m_i \alpha_i \tag{1.9}$$

We may further decompose[1] the root into components with roots in the A_{10} lattice and a vector orthogonal to it, z. We write,

$$\alpha_{11} = z - \lambda_8 \tag{1.10}$$

Where λ_i is the i'th fundamental weight of A_{10}, defined in relation to the simple roots of A_{10} by $< \lambda_i, \alpha_j >= \delta_{ij}$. Explicitly,

$$z = \frac{3}{11}(e_1 + \cdots + e_{11}) \tag{1.11}$$

Where $z^2 = -\frac{2}{11}$. The fact that the orthogonal component has an imaginary length is the sign of an indefinite algebra - from the Serre relations which define the Kac-Moody algebra we deduce that all roots in the algebra have length-squared less than or equal to two. It is the existence of a negative contribution to the root length that leads to an infinite algebra. It also allows us to sensibly decompose the infinite algebra into infinite copies of a finite algebra graded by the orthogonal imaginary component.

[1]We note that in the E_{11} literature m_{11}, the coefficient of α_{11}, is often referred to as the level of the decomposition and denoted l, but here, since we will consider decompositions characterised by more than one level we will continue using the notation m_{11}.

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We note that $\alpha_* = y + \frac{3}{2}z - \lambda_1$ in this decomposition. So that,

$$\beta = y + \left(\frac{3}{2} + m_{11}\right) z - \hat{\Lambda} \tag{1.12}$$

Where,

$$\hat{\Lambda} = \lambda_1 + m_{11}\lambda_8 - \sum_{i=1}^{10} m_i \alpha_i \equiv \sum_{i=1}^{10} p_i \lambda_i \tag{1.13}$$

$\hat{\Lambda}$ is a highest weight representation in the weight space of A_{10}, SL(11). By taking the inner product with the fundamental weight λ_j we find the coefficients of the simple roots in β,

$$m_j = \begin{cases} \frac{j}{11}(3m_{11} + A - 1) - B_j + 1, & j \leq 8 \\ \frac{j}{11}(-8m_{11} + A - 1) - B_j + 8m_{11} + 1, & j > 8 \end{cases} \tag{1.14}$$

Where we have defined the useful, integer expressions $A \equiv \sum_{i=1}^{10} ip_i$ and $B_j \equiv \sum_{i=1}^{j} ip_i + j\sum_{i>j}^{10} p_i$.The simple root coefficients must be positive integers. So a general solution is given by

$$m_{11} = \frac{1}{3}(-A + 1 + 11k) \tag{1.15}$$

Where k is an integer which is bounded from below by the condition that $m_1 \geq 1$, implying that $k \geq \sum p_i$. Let us explicitly realise the lower bound on k by rewriting k as

$$k \equiv \sum_{i=1}^{10} p_i + C \tag{1.16}$$

Where C is a constant greater than or equal to zero. Substituting this expression for k into the expression for the level, m_{11}, we have,

$$m_{11} = \frac{1}{3}\left(\sum_{i=1}^{10}(11 - i)p_i + 1 + 11C\right) \tag{1.17}$$

At each level m_{11} we will find representations of SL(11) described by the p_i and the new parameter C. We can now give a direct interpretation of C in terms of the blocks of antisymmetrised indices that appear on the SL(11) generators. We recall that the l_1 representation as a representation of SL(11) is formed from its highest weight, the translation generator, P_a, by its commutator action with the three-form generator, $R^{a_1a_2a_3}$. At level m_{11} the three form generator acts m_{11} times on the translation generator. Consequently we can express the number of indices, #, of any generator appearing at level m_{11} as

$$\# = 3m_{11} - 1 \tag{1.18}$$

By the same analysis one can see that number of indices on the generators of the adjoint representation of E_{11} is three times the level [20]. Substituting the expression for m_{11} we have,

$$\# = \sum_{i=1}^{10}(11 - i)p_i + 11C \tag{1.19}$$

The first term gives the index structure of an SL(11) tensor, that is the p_i control the blocks of indices in our generator of length less than eleven, while C, controls the number of blocks of eleven antisymmetrised indices. Blocks of eleven indices are proportional to the completely antisymmetric tensor, or volume form ϵ, in eleven dimensions, and correspond to trivial representations of SL(11). We therefore note that an interesting choice is to set $C = 0$ and so $k = \sum p_i$, if we do this we exclude from our algebra any generators containing the volume form, ϵ.

Substituting our expression for m_{11} into the simple root coefficient expressions above we have:

$$m_j = \begin{cases} jk - B_j + 1, & j \leq 8 \\ m_{11}(8-j) + jk - B_j + 1, & j > 8 \end{cases} \tag{1.20}$$

To reiterate we have solved the problem of finding which roots occur in the E_{12} lattice at a particular level m_{11} (with $m_* = 1$) in terms of an integer k which is bounded above and below. We observe that $m_1 \geq 1$ since we are considering irreducible representations meaning that roots in the algebra must have connected support on the Dynkin diagram[2] (and do not form a sub-algebra) and, by construction $m_* = 1$, therefore any root in the l_1 representation must have $m_1 \geq 1$. In our notation above, the variable k is bounded from below in terms of the weight coefficients of the A_{10} representation at each level:

$$k \geq \sum_{i=1}^{10} p_i \tag{1.21}$$

Consequently we find the generic root β corresponding to the l_1 representation of E_{11}:

$$\beta = e_* + \sum_{n=1}^{10} \left(k - \sum_{i=n}^{10} p_i \right) e_n + k e_{11} \tag{1.22}$$

Such that,

$$\beta^2 = \frac{1}{9}(8 - A^2 + 2A - 22k^2 - 22k + 4Ak) + \sum_{i=1}^{10} p_i B_i \tag{1.23}$$

The fact that $\beta^2 = 2, 0, -2\ldots$ gives an upper bound on k. An interesting class of roots occurs when we consider the lower bound for k, i.e. $k = \sum_i p_i$, which corresponds to generators having no blocks of eleven antisymmetrised indices, i.e. not including volume forms, ϵ. In this case,

$$\beta^2 = \frac{1}{9}\left[8 + \sum_{i=1}^{10} p_i^2 (11-i)(i-2) - 2\sum_{i=1}^{10} p_i(11-i) + 2\sum_{j>i} p_i p_j (11-j)(i-2) \right] \tag{1.24}$$

We recall that a putative root exists when $\beta^2 = 2, 0, -2\ldots$ is satisfied by the weight labels p_i for a given level m_{11}. At low levels one finds the translation generator, P_a, a two-form, Z^{ab}, and a five-form, Z^{abcde}. These are tensors which have the correct index structure to be

[2]That is, the non-zero coefficients of a root in E_{12} are all connected by the links of the Dynkin diagram when laid out on top of the diagram.

interpreted as the central charges of the supersymmetry algebra in eleven dimensions. This is quite surprising not least because we have been working solely with bosonic fields but also because the charge algebra is infinite and continues beyond the well-known supersymmetry charges. We list the low-level content derived in this section in table 25 in the appendix. The results of this computational problem were originally given in [5] and in the notation of this section, and to much higher levels, in [16].

1.2 The ten-dimensional IIA theory

We now carry out the reduction from the eleven-dimensional theory to the ten dimensional one, which has previously been carried out in [5], but as noted earlier we will use a different notation useful to our later computations. The IIA theory may be obtained from the eleven dimensional supergravity theory by dimensional reduction on a circle [23]. In terms of the algebra we delete a node of the A_{10} lattice that gives an A_9 lattice. For the l_1 representation this means the deletion of α_{10} in addition to the deletions of α_* and α_{11} given in section 1.1. The procedure is an extension of that carried out in the previous section. We denote,

$$\alpha_{10} = w - \nu_9 \tag{1.25}$$

Where ν_i are the fundamental weights of the A_9 lattice containing the roots $\alpha_1, \alpha_2, \ldots \alpha_9$. So that $w^2 = \frac{11}{10}$ and

$$w = \frac{1}{10}(e_1 + \cdots e_{10}) - e_{11} \tag{1.26}$$

In this decomposition we have,

$$\alpha_* = y + \frac{3}{2}z - \frac{1}{11}w - \nu_1 \tag{1.27}$$

$$\alpha_{11} = z - \nu_8 - \frac{8}{11}w \tag{1.28}$$

Therefore,

$$\beta = y + \left(\frac{3}{2} + m_{11}\right)z + \left(-\frac{1}{11} - \frac{8}{11}m_{11} + m_{10}\right)w - \Lambda \tag{1.29}$$

Where Λ is a highest weight in the A_9 algebra, specifically we have,

$$\Lambda = \nu_1 + m_{11}\nu_8 + m_{10}\nu_9 - \sum_{i=1}^{9} m_i\alpha_i \equiv \sum_{i=1}^{9} p_i\nu_i \tag{1.30}$$

Taking the inner product with ν_j we find,

$$m_j = \begin{cases} \frac{1}{10}(-A + 8m_{11} + 9m_{10} + 1), & j = 9 \\ \frac{j}{10}(A + 2m_{11} + m_{10} - 1) - B_j + 1, & j \leq 8 \end{cases} \tag{1.31}$$

Where $A = \sum_{i=1}^{9} ip_i$ and $B_j \equiv \sum_{i=1}^{j} ip_i + j\sum_{i>j}^{9} p_i$. To find integer coefficients we find solutions parameterised by two variables, q and k, and related to the two deleted nodes of

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the E_{11} part of the Dynkin diagram:

$$m_{11} = \frac{1}{2}(-A - q + 9k + 1), \tag{1.32}$$

$$m_{10} = q + k, \tag{1.33}$$

$$m_9 = m_{11} + q, \tag{1.34}$$

$$m_j = kj - B_j + 1, \qquad j \leq 8 \tag{1.35}$$

We note that $k \geq \sum_{i=1}^{9} p_i$ and $q \geq -k$. The solution corresponds to SL(10) tensors with $2m_{11} + m_{10} - 1$ indices, associated to m_{11} adjoint actions of $R^{a_1 a_2}$, m_{10} actions of R^b and one action of the translation generator, P_c in the algebra. Denoting the number of indices on a generator appearing in the algebra by #, we have,

$$\# \equiv 2m_{11} + m_{10} - 1 \tag{1.36}$$

$$= -A + 10k \tag{1.37}$$

$$= \sum_{i=1}^{9} (10 - i)p_i + 10C \tag{1.38}$$

Where in the last equality we have expressed the lower bound on the variable k by writing $k = \sum_{i=1}^{10} p_i + C$ where $C \geq 0$ is a constant. We see that C controls the blocks of ten anti-symmetrised indices appearing in the algebra, which are related to trivial representations of SL(10). By setting $C = 0$ and so $k = \sum_{i=1}^{10} p_i$ we neglect these trivial representations whose generators carry blocks of ten antisymmetrised indices. However for general k we have,

$$\beta = e_* + \sum_{n=1}^{9} \left(k - \sum_{i=n}^{10} p_i \right) e_n + k e_{10} + (m_{11} - m_{10}) e_{11} \tag{1.39}$$

And,

$$\beta^2 = 1 + 2q^2 + 2k^2 + A(q - k) - 6qk - q - k + \sum_{i=1}^{9} p_i B_i \tag{1.40}$$

As mentioned, the interesting set of solutions is given by considering $k = \sum_{i=1}^{9} p_i$ and let us rewrite $q = k + a$ where we have shifted q to simplify the notation and $a \geq -2k$ is an integer. In this case we have,

$$\beta^2 = 1 + a(2a - 1) + \sum_{i=1}^{9} p_i^2 (i - 2) + \sum_{i=1}^{10} p_i (a(i - 2) - 2) + 2 \sum_{j>i} p_i p_j (i - 2) \tag{1.41}$$

The results of this section are readily used to compute the charge algebra corresponding to the IIA theory and the weights at low levels are shown in table 26 in the appendix. Amongst the charges we find a scalar, Z, the charge of the fundamental string Z^a, the NS5 brane charge, $Z^{a_1 \dots a_5}$ and even forms $Z^{a_1 a_2}$, $Z^{a_1 \dots a_4}$, $Z^{a_1 \dots a_6}$, $Z^{a_1 \dots a_8}$ corresponding to the Ramond-Ramond brane charges of the D0, D2, D4, D6 and D8 branes of type IIA string theory.

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1.3 The ten-dimensional IIB theory

The charge algebra corresponding to the IIB theory, which has not been presented in the literature before, is obtained by deleting α_*, α_9 and α_{10} from the Dynkin diagram of E_{12} and finding representations of the A_9 algebra whose positive simple roots are $\alpha_1, \alpha_2, \ldots \alpha_8$ and α_{11}. In this decomposition we delete,

$$\alpha_* = y + x - \frac{1}{2}v - \mu_1 \tag{1.42}$$

$$\alpha_{10} = x - \frac{5}{2}v \tag{1.43}$$

$$\alpha_9 = v - \mu_8 \tag{1.44}$$

Where we denote by μ_i the fundamental weights of the A_9 Dynkin diagram dual to the roots $\{\alpha_1, \ldots \alpha_8, \alpha_{11}\}$. We have introduced the vectors,

$$x = \frac{1}{2}(e_1 + \cdots e_{10}) + e_{11}, \qquad x^2 = -\frac{1}{2} \tag{1.45}$$

$$v = \frac{1}{5}(e_1 + \cdots e_9) - \frac{1}{5}(e_{10}) + \frac{4}{5}e_{11}, \qquad v^2 = \frac{2}{5} \tag{1.46}$$

Now,

$$\beta = y + (1 + m_{10})x + \left(-\frac{1}{2} + m_9 - \frac{5}{2}m_{10} \right) z - \Lambda \tag{1.47}$$

Where,

$$\Lambda = \mu_1 + m_9\mu_8 - \sum_{i=1}^{8,11} m_i\alpha_i \equiv \sum_{i=1}^{8,11} p_i\mu_i \tag{1.48}$$

By taking inner products with the fundamental weights of the A_9 we find,

$$m_j = \begin{cases} \frac{1}{10}(-A + 8m_9 + 1), & j = 11 \\ \frac{j}{10}(A - 1 + 2m_9) - B_j + 1, & j \leq 8 \end{cases} \tag{1.49}$$

We parameterise the roots using $m_9 = \frac{1}{2}(-A + 1 + 10k)$ and find,

$$m_j = \begin{cases} m_9 - k, & j = 11 \\ jk - B_j + 1, & j \leq 8 \end{cases} \tag{1.50}$$

In this case we have $A = \sum_{i=1}^{8} ip_i + 9p_{11}$ and $B_j = \sum_{i=1}^{j} ip_i + j\sum_{i>j}^{8,11} p_i$. The solution gives tensors of A_{10} which have $2m_9 - 1$ indices, counting the number of actions (m_9) of the two form generator $Z^{a_1a_2}$ contracted with the translation generator P_a. Let us use # to denote the number of indices on an arbitrary generator appearing in the algebra, then,

$$\# \equiv 2m_9 - 1 \tag{1.51}$$

$$= -A + 10k \tag{1.52}$$

$$= \sum_{i=1}^{8}(10 - i)p_i + p_{11} + 10C \tag{1.53}$$

The last term counts blocks of ten antisymmetrised indices, corresponding to trivial representations of SL(10), and by setting $C = 0$, and thus $k = \sum_{i=1}^{8,11} p_i$, we disregard these representations. Note that in this case we have an SL(2) symmetry corresponding to the α_{10} root that was deleted and not directly attached to the A_9 gravity line of roots. This SL(2), given by a Dynkin diagram with a single node, has its own weight lattice. The fundamental weight in this A_1 diagram is $\nu \equiv \frac{1}{2}\alpha_{10}$. One can find the components of the vectors orthogonal to the A_9 lattice, x, v which are in the ν direction:

$$\frac{1}{x^2} < x, \nu >= -\frac{1}{2} \qquad \frac{1}{v^2} < v, \nu >= -1 \tag{1.54}$$

Therefore by writing the A_1 highest weight as $q\nu$ we find, taking the inner product with ν,

$$m_{10} = \frac{1}{2}(m_9 - q) \tag{1.55}$$

If we parameterise $q = m_9 - 2l$, we have,

$$m_{10} = l \tag{1.56}$$

Consequently,

$$\beta = e_* + \sum_{n=1}^{9}\left(k - \sum_{i=n}^{8,11} p_i\right)e_n + (l-k)e_{10} + (m_9 - k - l)e_{11} \tag{1.57}$$

And,

$$\beta^2 = 1 + 2l(l - m_9) + 10k^2 - 2kA + \sum_{i=1}^{8,11} p_i B_i \tag{1.58}$$

We write the root length squared in this form to illustrate the point that if $m_{10} = l$ gives a solution (i.e. root length squared less than or equal to two) then so does $m_{10} = m_9 - l$ which is due to the Weyl reflection perpendicular to α_{10}. For the special case $k = \sum_{i=1}^{8,11} p_i$ then,

$$\beta^2 = 1 + l(2l-1) + \sum_{i=1}^{8} p_i^2(10-i) + p_{11}^2 - \sum_{i=1}^{8} p_i(10-i) - p_{11} + 2\sum_{j>i}^{j=8} p_i p_j (10-j) + 2\sum_{i<11} p_i p_{11} \tag{1.59}$$

The results of this section are used to compute the charge algebra corresponding to the IIB theory to low-levels in table 27 of the appendix. Amongst the charges we find odd forms $Z^{a\alpha}$, $Z^{a_1\ldots a_3}$, $Z^{a_1\ldots a_5\alpha}$, $Z^{a_1\ldots a_7(\alpha\beta)}$ and $Z^{a_1\ldots a_9(\alpha\beta\gamma)}$ corresponding to the brane charges of the fundamental string and the D1 brane; the D3 brane; the D5 brane and the NS5 brane; D7, D9 branes as well as their S-dual charges of IIB string theory. The Greek indices $\alpha, \beta \ldots$ transform under the SL(2) symmetry.

2. Exotic charges

The U-duality transformations of toroidally compactified M-theory are the Weyl reflections of an E_n root lattice with different brane states being represented by weights of E_n [6, 7, 10,

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17, 18]. A group-theoretic approach to uncovering U-duality E_n multiplets was given in [8, 9] and we will recover aspects of this analysis. The weight vector corresponding to the brane solution encodes the tension of the BPS brane states and application of the Weyl reflections of E_n fills out the U-duality brane charge multiplets. Foe example the particle multiplet was discovered by encoding the well-known particle brane solution as a weight vector and then applying the U-duality transformations to it. The new solutions found under the action of all the combinations of the U-duality transformations completed the particle multiplet. In this construction the tension weight vector was introduced as an empirical tool. The brane multiplets containing the particle and string charges were recognised as fundamental representations of the E_n algebra. For example upon dimensional reduction to three dimensions (n=8) the particle multiplet has highest weight λ_1 (**248**) and the string multiplet λ_7 (**3875**), where λ_i are the fundamental weights of E_n. This work is discussed in detail in the original papers [6, 7, 10, 17, 18], but especially in the review [9].

In addition to the expected brane charges, coming from the dimensional reduction of charges associated to brane solutions in eleven dimensional supergravity, many exotic charges were also present in the U-duality charge multiplets whose higher dimensional origin was unknown. Exotic charges, in other words, are not derived from the dimensional reduction of the central charges appearing in the decomposition of the antiicommutator of the eleven dimensional supercharges [25],

$$\{Q_\alpha, \bar{Q}_\beta\} = \Gamma_{\alpha\beta} P_M + \frac{1}{2}\Gamma^{MN}_{\alpha\beta} Z_{MN} + \frac{1}{2}\Gamma^{MNPQR}_{\alpha\beta} Z_{MNPQR} \tag{2.1}$$

For example by dimensionally reducing P_M, Z_{MN}, Z_{MNPQR} to five dimensions one recovers the $\mathbf{6} \oplus \mathbf{15} \oplus \mathbf{6} = \mathbf{27}$ of SU(6) by assigning all the central charge indices to be internal indices. This agrees with the U-duality particle multiplet in five dimensions found in [6, 7, 10, 17, 18]. However the particle multiplet in four dimensions derived from the central charges of the eleven dimensional supersymmetry algebra reproduces only the $\mathbf{7} \oplus \mathbf{21} \oplus \mathbf{21} = \mathbf{49}$ of the degrees of freedom of the **56** of SU(6), which is the representation of the particle multiplet in four dimensions. The extra seven charges found in the particle multiplet in four dimensions are derived from the dimensional reduction of charges other than the usual central charges of the eleven dimensional superalgebra and are exotic charges. These extra states in $D = 4$ were associated to the compactification of the KK6 monopole, having charge $Z_{MNPQRST,R}$ which does not appear in the supersymmetry algebra, wrapping the T^7 in the compactification. In $D = 3$ many more exotic states were uncovered in the charge multiplets whose eleven dimensional origin was unclear.

The conjecture that the l_1 representation of E_{11} contains the full set of brane charges of M-theory gave an eleven-dimensional origin to these observations [5]. The Weyl group of the E_n sub-algebras, appearing upon dimensional reduction, being implicitly included from the outset. Furthermore the inclusion of the charge associated to the dual to gravity field in the eleven-dimensional algebra, the field giving rise to the KK6 monopole, from the outset explains the appearance of the exotic states appearing upon reduction to $D = 4$. The l_1 representation of E_{11} contains an infinite set of charges in addition to the central charges of the supergravity algebra and the charge of the KK6 monopole. It is these extra

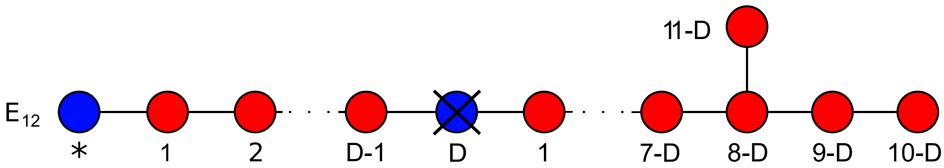

Figure 3: The decomposition of E_{12} to $A_{D-1} \otimes E_{11-D}$.

charges which give an eleven dimensional origin to the exotic charges appearing in the U-duality brane charge multiplets. These exotic charges are all associated to KK-branes, which we will consider further in section 3.

In this paper we will derive essentially the same tension formula that played a crucial role in uncovering the U-duality multiplets in [8, 9] but the expression given here will have a simple origin in the E_{11} conjecture. Compared to the original work uncovering the U-duality multiplets we will work in a reverse sense. First we will derive the brane charge multiplets from E_{11} and then second we will apply a tension formula to the results. Our aim will be to demonstrate the validity of the tension formula, but to do so we will first decompose the l_1 representation of E_{11} to various dimensions ($3 \leq D \leq 8$) and identify the brane charge multiplets. We note that the reduction to three dimensions of the l_1 representation of E_{11} has previously been carried out in [11] and the particle multiplet has been derived from l_1 representation in dimensions, D, where $3 \leq D \leq 8$ in [12] in all cases the results were in perfect agreement with the U-duality multiplet of charges.

In this section we extend the work of [11] and outline the decomposition of the l_1 algebra to arbitrary dimensions, $D < 11$, which is the algebraic equivalent of compactification on a (11-D)-torus and then apply the tension formula, which will be derived in section 3, to the charges that are found.

2.1 General decomposition

Let us give the decomposition of the l_1 representation of E_{11} in terms of its A_{D-1} and E_{11-D} sub-algebras. The details of this decomposition can be found in [11].

To find the l_1 representation of $A_{D-1} \otimes E_{11-D}$ we delete the node labelled D, in figure 3, to obtain, For example, to find representations of $A_4 \otimes E_6$ we delete the fifth node. The two deleted roots may be expressed in terms of vectors in the root space of E_{11} with components orthogonal to the vectors in the remaining root space after deletion of nodes $*$ and D. We carry out the decomposition to $A_{D-1} \otimes E_{11-D}$ by expressing the D'th root, α_D, in the form,

$$\alpha_D = -\nu + x, \qquad \nu = -\sum_{i=1, i \neq D}^{i=11} \lambda_i (A_{(E_{11})})_{iD} \tag{2.2}$$

Where $(A_{(E_{11})})_{iD}$ is the i, D'th component in the Cartan matrix of E_{11}, x is a vector in the root lattice orthogonal to all the remaining roots after the deletion of α_D, and now λ_i for $i = 1 \ldots D-1$ are the fundamental weights of A_{D-1}, and when $i = (D+1) \ldots 11$ they are fundamental weights of E_{11-D}. Denoting the fundamental weights of the decomposed

E_{11} by l_i we have,

$$\begin{aligned} l_D &= \frac{1}{x^2}x \\ l_i &= \lambda_i^{(r)} - \frac{< \nu, \lambda_i^{(r)} >}{x^2}x \end{aligned} \tag{2.3}$$

Where r is either 1, referring to the A_{D-1} sub-algebra, or 2, referring to the E_{11-D} sub-algebra. In this notation,

$$\alpha_* = -\lambda_1^{(1)} - \frac{< \lambda_1^{(1)}, \lambda_{D-1}^{(1)} >}{x^2}x + y \tag{2.4}$$

We now return to our consideration of E_{12}, a general root of which is given by,

$$\beta = m_*\alpha_* + m_D\alpha_D + \sum_{i=1, i\neq D}^{11} m_i\alpha_i \tag{2.5}$$

For a positive root, m_*, m_D and m_i are positive integers. Substituting our expressions for α_* and α_D, we obtain,

$$\beta = m_*y + \left(m_D - m_*\frac{< \lambda_1^{(1)}, \lambda_{D-1}^{(1)} >}{x^2}\right)x - \sum_{(r=1,2)} \Lambda^{(r)} \tag{2.6}$$

Where,

$$\Lambda^{(r)} = -\sum_{i=1, i\neq D}^{11} m_i^{(r)}\alpha_i^{(r)} + m_D\nu^{(r)} + m_*\lambda_1^{(r)}\delta_{(r,1)} \tag{2.7}$$

Adopting the notation $m_i^{(1)} = m_i$, $m_i^{(2)} = n_i$, $\lambda_i^{(1)} = \mu_i$ and $\lambda_i^{(2)} = \lambda_i$, we have,

$$\begin{aligned} \Lambda^{(1)} &= -\sum_{i=1}^{D-1} m_i\alpha_i^{(1)} + m_D\mu_{D-1} + m_*\mu_1 \equiv \sum_i^{D-1} q_i\mu_i \\ \Lambda^{(2)} &= -\sum_{i=1}^{11-D} n_i\alpha_i^{(2)} + m_D\lambda_1 \equiv \sum_i^{11-D} p_i\lambda_i \end{aligned} \tag{2.8}$$

Taking the inner product with μ_j and λ_j respectively we obtain expressions for the root coefficients, n_i and m_i,

$$\begin{aligned} A_{D-1}: &\qquad -m_j = \sum_i q_i < \mu_i, \mu_j > - m_D < \mu_{D-1}, \mu_j > - m_* < \mu_1, \mu_j > \\ E_{11-D}: &\qquad -n_j = \sum_i p_i < \lambda_i, \lambda_j > - m_D < \lambda_1, \lambda_j > \end{aligned} \tag{2.9}$$

We make use of formulae for the inner products of the fundamental weights of E_{11-D} derived in appendix A of reference [11],

$$\lambda_i = \begin{cases} \hat{\mu}_i + \frac{3i}{D-2}z, & i = 1, \ldots, 8-D \\ \hat{\mu}_i + \frac{(8-D)(11-D-i)}{D-2}z, & i = 9-D, 10-D \\ \frac{(11-D)}{(D-2)}z, & i = 11-D \end{cases} \tag{2.10}$$

These weights are derived by deleting the n'th node with respect to an E_n diagram, z is the vector in the root space corresponding to the linear independence of the n'th node and $\hat{\mu}_i$ are the weights of the A_{n-1} subalgebra, where $n = 11 - D$. We note that $z^2 = \frac{D-2}{11-D}$ and,

$$\lambda_1^2 = < \hat{\mu}_1, \hat{\mu}_1 > + \frac{9}{(D-2)(11-D)} = \frac{D-1}{D-2} \tag{2.11}$$

Consequently,

$$\alpha_D^2 = x^2 + \mu_{D-1}^2 + \lambda_1^2 = x^2 + \frac{D-1}{D} + \frac{D-1}{D-2} \tag{2.12}$$

Normalising $\alpha_D^2 = 2$ gives,

$$x^2 = \frac{-2}{D(D-2)} \tag{2.13}$$

We note that the vector x is given explicitly in our basis by:

$$x = \frac{1}{D}(e_1 + \cdots e_D) + \frac{1}{D-2}(e_{D+1} + \cdots e_{11}) \tag{2.14}$$

2.2 Rank p charges

We commence by looking for p-brane charges, that is rank p charges with all indices in the spacetime algebra, $Z^{a_1 \ldots a_p}$. This corresponds to a representation of the A_{D-1} sub-algebra with highest weight μ_{D-p}, where we recall that μ_i are the fundamental weights of A_{D-1}. The condition we must satisfy is that

$$\sum_i q_i \mu_i = \mu_{D-p} \tag{2.15}$$

A μ_{D-p} weight in the A_{D-1} sub-algebra of our decomposition leads to constraints upon the values to be taken by the root coefficient m_D.

We are interested in the decomposition of the l_1 representation of E_{11} and as such we take $m_* = 1$ and from the A_{D-1} equation in (2.9) we obtain,

$$-m_j = < \mu_{(D-p)}, \mu_j > - m_D < \mu_{(D-1)}, \mu_j > - < \mu_1, \mu_j > \tag{2.16}$$

$$= \begin{cases} (D-p-1) - \frac{j}{D}(D-p-1+m_D), & j \geq (D-p) \\ -1 + \frac{j}{D}(p+1-m_D), & j \leq (D-p) \end{cases} \tag{2.17}$$

Since $-m_j$ must be integer valued and negative we find a simple set of solutions for m_D having the form,

$$m_D = p + 1 + kD \tag{2.18}$$

Where k is a constant bounded from below because $m_1 \geq 1$, implying that $k \geq 0$. The solution corresponds in the algebra to m_D adjoint actions of a generator R^a with the translation generator P_c. The resulting generator in the algebra has $m_D - 1$ contravariant indices. Let us denote the indices by #, then,

$$\# \equiv m_D - 1 = p + kD \tag{2.19}$$

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The variable k corresponds to blocks of D antisymmetrised indices in the SL(D) algebra, and indicates the occurrence of trivial representations of SL(D). By setting $k = 0$ we may neglect these representations and simplify the algebra.

Having found a criterion for m_D corresponding to a p-brane charge in the A_{D-1} sub-algebra, we now turn our attention to restrictions on specific weights of the E_{11-D} sub-algebra consistent with the values of m_D corresponding to a rank p charge. We commence by finding conditions for representations of single fundamental weights of E_{11-D}, λ_i, for which we set $\sum_i p_i\lambda_i = \lambda_i$ in (2.9) and making use of equation (2.10) we find,

$$
\begin{aligned}
-n_j &= <\lambda_i, \lambda_j> - m_D <\lambda_1, \lambda_j> \\
&= \begin{cases}
i \leq 8-D \begin{cases} i - m_D + \frac{j}{D-2}(i - m_D), & j \leq 8-D, i \leq j \\ j - m_D + \frac{j}{D-2}(i - m_D), & j \leq 8-D, i \geq j \\ \frac{2(11-D-j)}{D-2}(i - m_D), & j = 9-D, 10-D \\ \frac{3}{D-2}(i - m_D), & \text{j=11-D} \end{cases} \\
i = 9-D, 10-D \begin{cases} -m_D + \frac{j}{D-2}(2(11-D-i) - m_D), \\ \qquad j \leq 8-D \\ \frac{11-D-j}{D-2}((8-D)^2 - i(6-D) - 2m_D), \\ \qquad j = 9-D, 10-D, i \leq j \\ \frac{1}{D-2}(4(11-D-i) - 2m_D), \\ \qquad j = 9-D, 10-D, i \geq j \\ \frac{1}{D-2}((8-D)(11-D-i) - 3m_D), \\ \qquad j = 11-D \end{cases} \\
i = 11-D \begin{cases} -m_D + \frac{j}{D-2}(3 - m_D), & j \leq 8-D \\ \frac{11-D-j}{D-2}(8 - D - 2m_D), & j = 9-D, 10-D \\ \frac{1}{D-2}(11 - D - 3m_D), & j = 11-D \end{cases}
\end{cases}
\end{aligned}
\tag{2.20}
$$

The simplest case with a solution is dependent upon the choice of fundamental weight λ_i and is

$$
m_D = \begin{cases} i + l(D-2), & i \leq 8-D \\ 2(11-D-i) + l(D-2), & i = 9-D, 10-D \\ 3 + l(D-2), & i = 11-D \end{cases} \tag{2.21}
$$

Where l is a positive integer or zero, which indicates trivial representations of E_{11-D}. Substituting the solution for m_D into the expression for the simple root coefficients, n_j, given above we have,

$$
\begin{aligned}
-n_j &= <\lambda_i,\lambda_j> - m_D <\lambda_1,\lambda_j> \qquad (2.22)\\
&= \begin{cases}
i \le 8-D \begin{cases} -l(D-2)-lj, & j\le 8-D, i\le j\\ j-i-l(D-2)-jl & j \le 8-D, i\ge j\\ -2l(11-D-j), & j=9-D,10-D\\ -3l, & j=11-D \end{cases}\\
i=9-D,10-D \begin{cases} -2(11-D-i)-l(D-2)-lj, \\ \qquad\qquad j\le 8-D\\ (11-D-j)((D-10)+(2l+i)), \\ \qquad\qquad j=9-D,10-D, i\le j\\ -2l, \\ \qquad\qquad j=9-D,10-D,i\ge j\\ (-11+D+i-3l), \\ \qquad\qquad j=11-D \end{cases}\\
i=11-D \begin{cases} -m_D-jl, & j\le 8-D\\ (11-D-j)(-1-2l), & j=9-D,10-D\\ -1-3l, & j=11-D \end{cases}
\end{cases}
\end{aligned}
$$

The dependence on the parameter l is made manifest and we see that $l \ge 0$. Notice that when E_{11-D} is decomposed into representations of SL$(11-D)$ by deleting node $(11-D)$, the parameter l indicates blocks of $11-D$ antisymmetrised indices. Explicitly, deletion of node $(11-D)$ together with the deletion of node D leads to tensors with $3n_{11-D} - m_D$ which we denote # so that,

$$
\# = \begin{cases} -i + l(11-D), & i \le 8-D\\ (11-D-i)+l(11-D) & i = 9-D, 10-D\\ l(11-D) & i=11-D \end{cases} \qquad (2.23)
$$

This allows us to see that l controls the appearance of blocks of $11-D$ antisymmetrised indices. By setting $l=0$ we disregard the trivial representations of E_{11-D}. We note the subtlety that a trivial representation in the E_{11-D} algebra, after the action of the local sub-group, may give rise to non-trivial representations in the spacetime SL(D) algebra and a non-trivial representation of E_{11-D}, we will discuss this possibility in section 2.6.

We choose values of m_D which give a rank p charge in the A_{D-1} sub-algebra and then read off the value of the corresponding fundamental weight in the E_{11-D} sub-algebra. This amounts to equating our two conditions for m_D, equations (2.18) and (2.21),

$$
p = \begin{cases} i + l(D-2) - kD - 1, & i \le 8-D\\ 2(11-D-i) + l(D-2) - kD - 1, & i = 9-D, 10-D\\ 2 + l(D-2) - kD, & i = 11-D \end{cases} \qquad (2.24)
$$

In particular for $l = k = 0$ we find the charge content indicated in table 1, where the charges are indicated by SL$(D) \otimes$ SL$(11-D)$ tensors. We use the index a_i to indicate an index in the spacetime associated to the weight of A_{D-1} being considered, and the index j_i

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Index of λ_i	Weight of $E_{11-D} \otimes A_{D-1}$	Highest Weight Charge
$i \leq 8 - D$	$\lambda_i \otimes \mu_{D-i}$	$Z^{a_1 \ldots a_{i-1} j_1 \ldots j_{11-D-i}}$
$i = 9 - D$	$\lambda_{9-D} \otimes \mu_{D-3}$	$Z^{a_1 a_2 a_3 j_1 j_2}$
$i = 10 - D$	$\lambda_{10-D} \otimes \mu_{D-1}$	$Z^{a_1 j_1}$
$i = 11 - D$	$\lambda_{11-D} \otimes \mu_{D-2}$	$Z^{a_1 a_2 j_1 j_2 j_3}$

Table 1: Charges associated to fundamental weights of E_{11-D} upon dimensional reduction indicated by $\mathrm{SL}(D) \otimes \mathrm{SL}(11-D)$ tensors

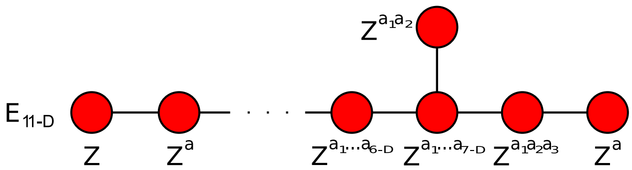

Figure 4: The Dynkin diagram of E_{11-D}.

to indicate internal coordinates coming from the representation of E_{11-D} that the charge transforms under. The fundamental weights of E_{11-D} belong to a given representation of $\mathrm{SL}(11-D)$, so we may label them by the highest weight of their $\mathrm{SL}(11-D)$ representation. For example, if we consider the representation of E_{11-D} whose highest weight is λ_{11-D} it belongs to the representation of SL_{11-D} with highest weight $Z^{(9-D)(10-D)(11-D)}$. In table 1 it is the highest weight of $\mathrm{SL}(11-D)$ that we have used to indicate the E_{11-D} representation.

For representations of E_{11-D} the internal indices in the decomposition to $\mathrm{SL}(D) \otimes \mathrm{SL}(11-D)$ tensors will vary throughout the multiplet, but the spacetime indices will remain unaltered. Let us consider the example of the particle charge multiplet in $D = 3$ which is the first fundamental representation of E_8 with highest weight, λ_1. The **248** has $\mathrm{SL}(D) \otimes \mathrm{SL}(11-D)$ tensors:

$$\begin{array}{cccc} P_j(8), & Z^{j_1 j_2}(28), & Z^{j_1 \cdots j_5}(56), & Z^{j_1 \cdots j_7, k}(63), \\ Z^{j_1 \cdots j_8}(1), & Z^{j_1 \cdots j_3}(56), & Z^{j_1 \cdots j_6}(28), & Z^j(1) \end{array} \tag{2.25}$$

It is convenient to indicate the tensor charge associated to just the highest weight in the representation, it is this that is indicated by the charges in table 1. From table 1 we read that such a charge $Z^{a_1 j_1}$ is associated to the (10-D)'th fundamental representation (with highest weight λ_{10-D}) of E_{11-D}, and so on for the other charge multiplets. It may be useful to associate the various charge multiplets associated to a fundamental weight of $E_{11=D}$ with their nodes on the E_{11-D} Dynkin diagram,

For the case of $D = 3$ [11], the charges associated to the fundamental weights of nodes $11 - D, 10 - D$ and 1 of E_{11-D} are the highest weights of the membrane, string and particle multiplets [9]. The set of charges up to rank $D - 1$ where D ranges from three to eight derived from the l_1 multiplet is shown in table 2. Many of the charge multiplets indicated therein are associated to representations of E_{11-D} whose highest weight is a sum

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D	G	Z	Z^a	$Z^{a_1a_2}$	$Z^{a_1\ldots a_3}$	$Z^{a_1\ldots a_4}$	$Z^{a_1\ldots a_5}$	$Z^{a_1\ldots a_6}$	$Z^{a_1\ldots a_7}$
8	SU(3) ⊗ SU(2)	$(\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{1},\mathbf{3})$ $(\mathbf{8},\mathbf{1})$ $(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$ $(\mathbf{6},\mathbf{2})$	$(\mathbf{6},\mathbf{1})$ $(\mathbf{18},\mathbf{1})$ $(\mathbf{3},\mathbf{1})$ $(\mathbf{6},\mathbf{1})$ $(\mathbf{3},\mathbf{3})$
7	SU(5)	**10**	**5**	**5**	**10**	**24** **1**	**40** **15** **10**	**70** **50** **45** **5**	- - - -
6	SO(5, 5)	**16**	**10**	**16**	**45** **1**	**144** **16**	**320** **126** **120**	- - -	- - -
5	E_6	**27**	**27**	**78** **1**	**351** **27**	**1728** **351** **27**	- - -	- - -	- - -
4	E_7	**56**	**133** **1**	**912** **56**	**8645** **1539** **133** **1**	- - - -	- - - -	- - - -	- - - -
3	E_8	**248** **1**	**3875** **248** **1**	**147250** **30380** **3875** **248** **1**	- - - - -	- - - - -	- - - - -	- - - - -	- - - - -

Table 2: Charge multiplet representations of the group, G, from the l_1 representation of E_{11} in $3 \leq D \leq 8$.

of fundamental weights. We deal with these cases in section 2.6, where we extract the full content of table 2 from the l_1 algebra.

2.3 The particle multiplet

We identified the highest weight of this multiplet above, and we now discuss the identification of the weights of the l_1 representation associated to particle, or zero brane, charges in detail. The particle charge multiplet will consist of objects having no spacetime indices (a_i), and a set of internal indices (j_i). From table 1 we see that this charge corresponds to the representation with highest weight ($\lambda_1 \otimes \mu_{D-1}$), having a highest weight charge indicated by an SL$(11-D)$ tensor, $Z^{j_1\cdots j_{10-D}}$, i.e. it has only internal indices and it transforms in the first fundamental representation of the E_{11-D} sub-group.

To find the roots explicitly we recall that this is the representation whose highest weight is the first fundamental weight of E_{11-D}, so that in $D = 3$ it is the **248** of E_8, in $D = 4$ it is the **56** of E_7 and so on. In section 2.2 the condition on the highest weight representation

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E_{12} root	Charge	Dimension of SL(8) tensor	Mass (highest weight)
$(1^5, 0^7)$	P_j	8	$\frac{1}{R_4}$
$(1^9, 0^2, 1)$	$Z^{j_1 j_2}$	28	$\frac{R_{10}R_{11}}{l_p^3}$
$(1^7, 2, 3, 2, 1, 2)$	$Z^{j_1 \dots j_5}$	56	$\frac{R_7 \dots R_{11}}{l_p^6}$
$(1^5, 2, 3, 4, 5, 6, 4, 2, 3)$	$Z^{j_1 \dots j_8}$	1	$\frac{V}{l_p^9}$
$(1^5, 2, 3, 4, 5, 3, 1, 3)$	$Z^{j_1 \dots j_7, k}$	63	$\frac{V R_{11}}{R_4 l_p^9}$
$(1^4, 2, 3, 4, 5, 6, 4, 2, 4)$	$Z^{j_1 \dots j_8, k_1 \dots k_3}$	56	$\frac{V R_9 \dots R_{11}}{l_p^{12}}$
$(1^4, 2, 3, 5, 7, 9, 6, 3, 5)$	$Z^{j_1 \dots j_8, k_1 \dots k_6}$	28	$\frac{V R_6 \dots R_{11}}{l_p^{15}}$
$(1^4, 3, 5, 7, 9, 11, 7, 3, 6)$	$Z^{j_1 \dots j_8, k}$	8	$\frac{V^2 R_{11}}{l_p^{18}}$

Table 3: The particle charge multiplet in $D = 3$/The **248** of E_8.

of SL(D) has been written in terms of p, where the charge multiplet's highest weight is the charge associated to a p-brane. From equation (2.16), with $p = 0$ for the particle charge multiplet we find that,

$$m_j = 1 \qquad j = 1, \dots D - 1 \tag{2.26}$$

From equation 2.18 we have $m_D = 1$. Since we are considering the l_1 representation $m_* = 1$ then the particle multiplet contains roots with root coefficients $(1^{D+1}, m_{D+1}, \dots m_{11})$. We use the superscript notation as a shorthand to indicate that a number of sequential roots share the same coefficient, e.g.

$$(1^{D+1}, m_{D+1}, \dots m_{11}) \equiv \alpha_* + \alpha_1 + \dots + \alpha_D + \sum_{i=D+1}^{11} m_i \alpha_i \tag{2.27}$$

The simple root coefficients $\{m_{i>D}\}$ correspond to the weights of the first fundamental representation of E_{11-D} which we will identify. For the reduction to D dimensions we should find the particle multiplet being made up of roots whose first $D+1$ root coefficients are 1's. The most complicated case is that of the reduction to $D = 3$ which has been studied already in [11], and the l_1 representations have been shown to form the **248** of E_8. In addition to this condition we must identify the root vectors of the fundamental representation of E_8 amongst these roots. By finding all roots of the form $(1^{D+1}, m_{D+1}, \dots m_{11})$ we identify the $\mathbf{248} \oplus \mathbf{1}$ indicated in table 2. To distinguish the roots of **248** we must identify the root vectors of the fundamental **248** amongst the candidate roots. In more complicated cases to follow this process will be less trivial than in the present example. From the l_1 representation of E_{11} we can read off the charges in the particle multiplet and these are listed in table 3. In section 3 of this paper we will give a formula, equation (3.6), relating a root in the E_{12} lattice with a mass. In table 3 we give the result of applying this formula to the roots in the table, the resulting masses exactly match those found by U-duality [8, 9].

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E_{12} root	Charge	Dimension of SL(7) tensor	Mass
$(1^5, 0^7)$	P_j	7	$\frac{1}{R_5}$
$(1^9, 0, 0, 1)$	$Z^{j_1 j_2}$	21	$\frac{R_{10}R_{11}}{l_p^3}$
$(1^7, 2, 3, 2, 1, 2)$	$Z^{j_1 \cdots j_5}$	21	$\frac{R_7 \ldots R_{11}}{l_p^6}$
$(1^5, 2, 3, 4, 5, 3, 1, 3)$	$Z^{j_1 \cdots j_7, k}$	7	$\frac{V R_{11}}{l_p^9}$

Table 4: The particle charge multiplet in $D = 4$/The **56** of E_7.

E_{12} root	Charge	Dimension of SL(6) tensor	Mass
$(1^6, 0^6)$	P_j	6	$\frac{1}{R_6}$
$(1^9, 0, 0, 1)$	$Z^{j_1 j_2}$	15	$\frac{R_{10}R_{11}}{l_p^3}$
$(1^7, 2, 3, 2, 1, 2)$	$Z^{j_1 \cdots j_5}$	6	$\frac{R_7 \ldots R_{11}}{l_p^6}$

Table 5: The particle charge multiplet in $D = 5$/The **27** of E_6.

In table 3 we have used V to indicate $R_{D+1}R_{D+2}\ldots R_{11}$ indicating the volume of the internal space. From table 3 we can identify generators which commute with each other to form generators proportional to multiple copies of the volume form($\epsilon^{\mu_1 \cdots \mu_{11-D}}$). We call such pairs of generators dual. The Young tableaux of a generator and its dual may be combined to form a rectangular tableau whose height is the dimension of the internal space. Such a Young tableau is proportional to multiple volume tensors. For example, in table 3 the generators at levels $0, 1, 2$ are dual to those at $4, 5, 6$, and those at level 3 are self-dual, that is

$$[P_j, Z^{l_1 \ldots l_8, k_1 \ldots k_8, j}] \propto \epsilon^2, \qquad [Z^{l_1 l_2}, Z^{j_1 \ldots j_8, k_1 \ldots k_6}] \propto \epsilon^2, \qquad [Z^{j_1 \ldots j_5}, Z^{l_1 \ldots l_8, k_1 \ldots k_3}] \propto \epsilon^2$$
$$[Z^{j_1 \ldots j_8}, Z^{k_1 \ldots k_8}] \propto \epsilon^2, \qquad [Z^{j_1 \ldots j_7, k}, Z^{l_1 \ldots l_7, m}] \propto \epsilon^2 \tag{2.28}$$

We are identifying a bilinear Casimir for the representation. By using the mass formula (to be derived in section 3) one can see that the mass associated to a generator, $\mathcal{M}$, and the mass associated to its dual generator, $\mathcal{M}'$, multiply to give an invariant squared mass, $\mathcal{M}\mathcal{M}' = \mathcal{M}^2$, for the representation. For the **248** shown in table 3 the invariant mass squared is:

$$\mathcal{M}^2_{\mathbf{248}} \sim \frac{V^2}{l_p^{18}} \tag{2.29}$$

Let us look at the reduction to $D = 4$ where the particle multiplet should belong to representations of E_7. From tables of E_{12} roots [16], we find the l_1 content listed in table 4. By identifying the dual generators in table 4 one can associate an invariant mass squared to the multiplet, $\mathcal{M}^2_{\mathbf{56}} \sim \frac{V}{l_p^9}$. In the reduction to $D = 5$ we look for the **27** of E_6, and the corresponding roots are in table 5. For the reduction to $D = 6$ we look for the **16** of SO(5, 5). From the l_1 we find the states of table 6. We repeat the process for $D = 7$ where we look for the **10** of SU(5), the appropriate roots are listed in table 7. The formulae used in this section are even robust up to $D = 8$, where we identify the $(\mathbf{3}, \mathbf{2})$ of $\mathrm{SU}(3) \otimes \mathrm{SU}(2)$

E_{12} root	Charge	Dimension of SL(5) tensor	Mass
$(1^7, 0^5)$	P_j	5	$\frac{1}{R_7}$
$(1^9, 0, 0, 1)$	$Z^{j_1 j_2}$	10	$\frac{R_{10} R_{11}}{l_p^3}$
$(1^7, 2, 3, 2, 1, 2)$	Z	1	$\frac{R_7 \ldots R_{11}}{l_p^6}$

Table 6: The particle charge multiplet in $D = 6$/The **16** of SO(5, 5).

E_{12} root	Charge	Dimension of SL(4) tensor	Mass
$(1^8, 0^3)$	P_j	4	$\frac{1}{R_8}$
$(1^9, 0, 0, 1)$	$Z^{j_1 j_2}$	6	$\frac{R_{10} R_{11}}{l_p^3}$

Table 7: The particle charge multiplet in $D = 7$/The **10** of SL(5)

D	Z	Z^j	$Z^{j_1 j_2}$	$Z^{j_1 \cdots j_3}$	$Z^{j_1 \cdots j_4}$	$Z^{j_1 \cdots j_5}$	$Z^{j_1 \cdots j_6}$	$Z^{j_1 \cdots j_7}$
3	$64^{(3)}$	$8^{(6)}$	$28^{(1)}$	$56^{(4)}$		$56^{(2)}$	$28^{(5)}$	$8^{(0)}$
4		$7^{(3)}$	$21^{(1)}$			$21^{(2)}$	$7^{(0)}$	- - -
5			$15^{(1)}$			$6^{(0)}$ $6^{(2)}$	- -	- -
6	$1^{(2)}$		$10^{(1)}$		$5^{(0)}$	- - -	- - -	- - -
7			$6^{(1)}$	$4^{(0)}$	- -	- -	- -	- -
8			$3^{(0)}$ $3^{(1)}$	- -	- -	- -	- -	- -

Table 8: The particle charge multiplets in $3 \leq D \leq 8$.

contained in the l_1 representation from the roots $(1^9, 0, 0, 1)$ and $(1^9, 0^3)$, associated to generators $Z^{j_1 j_2}$ and P_j, both having dimension three.

We summarise the appearance of the particle multiplet from the l_1 representation in table 8. The table shows the dimension of $\mathrm{SL}(11 - D)$ tensors together with the level the generator appears at in the decomposition, so that $X^{(l)}$ indicates a tensor carrying X degrees of freedom appearing in the decomposition at level l. The columns indicate the number of indices in the charge modulo blocks of $11 - D$ indices.

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2.4 The string multiplet

The string multiplet contains charges with one spacetime index. From table 1 we see that such charges are associated to the representation of E_{11-D} with highest weight λ_{10-D}. In $D = 3$ this is the **3875** of E_8, in $D = 4$ it is the **133** of E_7 and so on. From section 2.2. we recall that the string multiplet corresponds to 1-brane charges, by putting $p = 1$ and $k = 0$ in equation (2.18) we find the simple root coefficient $m_D = 2$ for string charges. From equation (2.16), with $p = 1$ for the string, we find that,

$$m_j = 1 \qquad j = 1, \dots D - 1 \tag{2.30}$$

The candidate roots in the E_{12} lattice corresponding to the string multiplet have simple root coefficients $(1^D, 2, m_{D+1}, \dots m_{11})$. The $m_{i>D}$ take values which vary according to the weight vector of the representation of E_{11-D} with highest weight λ_{10-D}.

Using equation (2.20) we find the root in the E_{12} lattice corresponding to the weight μ_{10-D} of E_{11-D} ($p_{10-D} = 1$ and all other $p_i = 0$), that is the highest weight of the representation we are seeking to identify, is $(1^D, 2^{9-D}, 0, 0, 1)$. This root does not appear in the l_1 representation however, as discussed in [11], it is related to the root $(1^9, 0, 0, 1)$ in the l_1 representation, with charge $Z^{8.11}$ by a series commutators with the generators $K^D{}_{D+1}$, $K^{D+1}{}_{D+2}$, ... $K^7{}_8$, giving the charge $Z^{D.11}$. This charge has one compact index (j=11) and one non-compact index (a=D), and so is a component of a string charge Z^{aj}. We can also apply the generators $K^a{}_j$ to the roots of the particle multiplet and find string charges in a similar manner. To commence we rediscover the E_{12} roots of the string multiplet in $D = 3$. These roots were originally given in [11] and we list them in table 9 together the associated masses given by equation (3.6) which gives expressions in agreement with the string multiplet listed explicitly in appendix B of [9]. The procedure of searching the low levels of the l_1 representation for roots with root coefficients of the form $(1^3, 2, m_4, \dots m_{11})$ reproduces the $\mathbf{3875} \oplus \mathbf{248} \oplus \mathbf{1}$ of E_8, from which the **3875** can be identified. It is a simple matter to pair the dual generators in the string multiplet, to find an invariant mass squared for the multiplet:

$$\mathcal{M}^2_{\mathbf{3875}} \sim \frac{R_a^2 V^4}{l_p^{36}} \tag{2.31}$$

We find the string multiplet in $D = 4$ has E_{12} roots with root coefficients $(1^4, 2, m_5 \dots m_{11})$. In $D = 4$ we look for the **133** of E_7 and the corresponding roots from the l_1 are listed in table 10. The invariant mass squared for the multiplet is,

$$\mathcal{M}^2_{\mathbf{133}} \sim \frac{R_a^2 V^2}{l_p^{18}} \tag{2.32}$$

In $D = 5$ we look for the **27** of E_6. This corresponds to roots of the form $(1^5, 2, m_6 \dots m_{11})$, which we list in table 11. In $D = 6$ we look for the **10** of SO(5, 5). This corresponds to roots of the form $(1^6, 2, m_7 \dots m_{11})$, which we list in table 12. The associated invariant mass squared for the representation in table 12 is

$$\mathcal{M}^2_{\mathbf{10}} \sim \frac{R_a^2 V}{l_p^{9}} \tag{2.33}$$

E_{12} Root	Charge	Dimension of SL(8) tensor	Mass
$(1^3, 2^6, 0, 0, 1)$	Z^{a,j_1}	8	$\frac{R_3 R_{11}}{l_p^3}$
$(1^3, 2^5, 3, 2, 1, 2)$	$Z^{a,j_1 \ldots j_4}$	70	$\frac{R_3 R_8 \ldots R_{11}}{l_p^6}$
$(1^3, 2^3, 3, 4, 5, 3, 1, 3)$	$Z^{a,j_1 \ldots j_6,k}$	216	$\frac{R_3 R_6 \ldots R_{10} R_{11}^2}{l_p^9}$
$(1^3, 2^2, 3, 4, 5, 6, 4, 2, 3)$	$Z^{a,j_1 \ldots j_7}$	8	$\frac{R_3 R_5 \ldots R_{11}}{l_p^9}$
$(1^3, 2^2, 3, 4, 5, 6, 4, 2, 4)$	$Z^{a,j_1 \ldots j_7,k_1 \ldots k_3}$	420	$\frac{R_3 V R_9 R_{10} R_{11}}{R_4 l_p^{12}}$
$(1^3, 2, 3, 4, 5, 6, 7, 4, 1, 4)$	$Z^{aj_1 \ldots j_8,k_1 k_2}$	28	$\frac{R_3 V (R_{10} R_{11})^2}{l_p^{12}}$
$(1^3, 2, 3, 4, 5, 6, 7, 4, 2, 4)$	$Z^{aj_1 \ldots j_8,(k_1,l_1)}$	36	$\frac{R_3 V R_{11}^2}{l_p^{12}}$
$(1^3, 2^2, 3, 5, 7, 9, 6, 3, 5)$	$Z^{aj_1 \ldots j_7,k_1 \ldots k_6}$	168	$\frac{R_3 V R_6 \ldots R_{11}}{R_4 l_p^{15}}$
$(1^3, 2, 3, 4, 5, 7, 9, 6, 3, 5)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_5}$	56	$\frac{R_3 V R_7 \ldots R_{11}}{l_p^{15}}$
$(1^3, 2, 3, 4, 5, 6, 8, 5, 2, 5)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_4,l_1}$	504	$\frac{R_3 V R_8 \ldots R_{10} R_{11}^2}{l_p^{15}}$
$(1^3, 2, 4, 6, 8, 10, 12, 8, 4, 6)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8}$	1	$\frac{R_3 V^2}{l_p^{18}}$
$(1^3, 2, 3, 5, 7, 9, 11, 7, 3, 6)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_7,l_1}$	63	$\frac{R_3 V^2 R_{11}}{R_4 l_p^{18}}$
$(1^3, 2, 3, 4, 6, 8, 10, 6, 3, 6)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_6,l_1 l_2}$	720	$\frac{R_3 V^2 R_{10} R_{11}}{R_4 R_5 l_p^{18}}$
$(1^3, 2, 3, 5, 7, 9, 11, 7, 3, 6)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_7,l_1}$	63	$\frac{R_3 V^2 R_{11}}{R_4 l_p^{18}}$
$(1^3, 2, 3, 5, 7, 9, 12, 8, 4, 7)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_7,l_1 \ldots l_4}$	504	$\frac{R_3 V^2 R_8 \ldots R_{11}}{R_4 l_p^{21}}$
$(1^3, 2, 4, 6, 8, 10, 12, 8, 4, 7)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 \ldots l_3}$	56	$\frac{R_3 V^2 R_9 \ldots R_{11}}{l_p^{21}}$
$(1^3, 2, 4, 6, 8, 10, 12, 7, 3, 7)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 l_2,m_1}$	168	$\frac{R_3 V^2 R_{10} R_{11}^2}{l_p^{21}}$
$(1^3, 2, 3, 6, 9, 12, 15, 10, 5, 8)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_7,l_1 \ldots l_7}$	36	$\frac{R_3 V^3}{R_4^2 l_p^{24}}$
$(1^3, 2, 4, 6, 9, 12, 15, 10, 5, 8)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 \ldots l_6}$	28	$\frac{R_3 V^3}{R_4 R_5 l_p^{24}}$
$(1^3, 2, 4, 6, 8, 11, 14, 9, 4, 8)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 \ldots l_5,m_1}$	420	$\frac{R_3 V^3 R_{11}}{R_4 R_5 R_6 l_p^{24}}$
$(1^3, 2, 5, 8, 11, 14, 17, 11, 5, 9)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 \ldots l_8,m_1}$	8	$\frac{R_3 V^3 R_{11}}{l_p^{27}}$
$(1^3, 2, 4, 7, 10, 13, 16, 10, 5, 9)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 \ldots l_7,m_1 m_2}$	216	$\frac{R_3 V^3 R_{10} R_{11}}{R_4 l_p^{27}}$
$(1^3, 2, 5, 8, 11, 14, 18, 12, 6, 10)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 \ldots l_8,m_1 \ldots m_4}$	70	$\frac{R_3 V^3 R_8 \ldots R_{11}}{l_p^{30}}$
$(1^3, 2, 5, 9, 13, 17, 21, 14, 7, 11)$	$Z^{aj_1 \ldots j_8,k_1 \ldots k_8,l_1 \ldots l_8,m_1 \ldots m_7}$	8	$\frac{R_3 V^4}{R_4 l_p^{33}}$

Table 9: The string charge multiplet in $D = 3$/The **3875** of E_8.

E_{12} root	Charge	Dimension of SL(7) tensor	Mass
$(1^4, 2^5, 0, 0, 1)$	Z^{aj_1}	7	$\frac{R_4 R_{11}}{l_p^3}$
$(1^4, 2^4, 3, 2, 1, 2)$	$Z^{aj_1 \ldots j_4}$	35	$\frac{R_4 R_8 \ldots R_{11}}{l_p^6}$
$(1^4, 2^2, 3, 4, 5, 3, 1, 3)$	$Z^{aj_1 \ldots j_6,k}$	49	$\frac{R_4 V R_{11}}{R_5 l_p^9}$
$(1^4, 2, 3, 4, 5, 6, 4, 2, 4)$	$Z^{aj_1 \ldots j_7,k_1 \ldots k_3}$	35	$\frac{R_4 V R_9 \ldots R_{11}}{l_p^{12}}$
$(1^4, 2, 3, 5, 7, 9, 6, 3, 5)$	$Z^{aj_1 \ldots j_7,k_1 \ldots k_6}$	7	$\frac{R_4 V^2}{R_5 l_p^{15}}$

Table 10: The string charge multiplet in $D = 4$/The **133** of E_7.

In $D = 7$ we look for the **5** of SL(5). This corresponds to roots of the form $(1^7, 2, m_8 \ldots m_{11})$, which we list in table 13. In $D = 8$ we find the $(\mathbf{1}, \mathbf{3})$ of SU(2) × SU(3). This corresponds to roots of the form $(1^8, 2, m_9 \ldots m_{11})$, and we find only the root with

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E_{12} root	Charge	Dimension of SL(6) tensor	Mass
$(1^5, 2^4, 0, 0, 1)$	Z^{aj_1}	6	$\frac{R_5 R_{11}}{l_p^3}$
$(1^5, 2^3, 3, 2, 1, 2)$	$Z^{aj_1 \ldots j_4}$	15	$\frac{R_5 R_8 \ldots R_{11}}{l_p^6}$
$(1^5, 2, 3, 4, 5, 3, 1, 3)$	$Z^{aj_1 \ldots j_6, k_1}$	6	$\frac{R_5 V R_{11}}{l_p^9}$

Table 11: The string charge multiplet in $D = 5$/The **27** of E_6.

E_{12} root	Charge	Dimension of SL(5) tensor	Mass
$(1^6, 2^3, 0, 0, 1)$	$Z^{a,j}$	5	$\frac{R_6 R_{11}}{l_p^3}$
$(1^6, 2^2, 3, 2, 1, 2)$	$Z^{aj_1 \ldots j_4}$	5	$\frac{R_6 R_8 \ldots R_{11}}{l_p^6}$

Table 12: The string charge multiplet in $D = 6$/The **10** of SO(5, 5).

E_{12} root	Charge	Dimension of SL(4) tensor	Mass
$(1^7, 2^2, 0, 0, 1)$	$Z^{a,j}$	4	$\frac{R_7 R_{11}}{l_p^3}$
$(1^7, 2, 3, 2, 1, 2)$	$Z^{a,j_1 \ldots j_4}$	1	$\frac{R_7 V}{l_p^6}$

Table 13: The string charge multiplet in $D = 7$/The **5** of SL(5).

D	Z^a	Z^{aj}	$Z^{aj_1j_2}$	$Z^{aj_1 \ldots j_3}$	$Z^{aj_1 \ldots j_4}$	$Z^{aj_1 \ldots j_5}$	$Z^{aj_1 \ldots j_6}$	$Z^{aj_1 \ldots j_7}$
3		$8^{(1)}$			$70^{(2)}$			$224^{(3)}$
			$484^{(4)}$			$728^{(5)}$		
	$847^{(6)}$			$728^{(7)}$			$484^{(8)}$	
		$224^{(9)}$			$70^{(10)}$			$8^{(11)}$
4		$7^{(1)}$			$35^{(2)}$			-
	$49^{(3)}$			$35^{(4)}$			$7^{(5)}$	-
5		$6^{(1)}$			$15^{(2)}$		-	-
		$6^{(3)}$					-	-
6		$5^{(1)}$			$5^{(2)}$	-	-	-
7		$4^{(1)}$			-	-	-	-
	$1^{(2)}$				-	-	-	-
8		$3^{(1)}$		-	-	-	-	-

Table 14: The string charge multiplets in $3 \leq D \leq 8$.

simple root coefficients $(1^8, 2, 0, 0, 1)$, corresponding to a charge Z^{aj_1} of dimension 3 and tension $\frac{R_8 R_{11}}{l_p^3}$. The appearance of the string multiplet in the l_1 representation is summarised for $3 \leq D \leq 8$ in table 14.

In addition to the string and particle multiplets, we may also find multiplets associated with the membrane charge, a threebrane charge, a fourbrane charge and other charges up to that of a $D-1$-brane, as discussed in [8, 9]. One might also find a fivebrane multiplet whose highest weight is $\mu_{D-5} \otimes 2\lambda_{11-D}$, indeed one may find multiplets corresponding to a variety of exotic charges whose interpretation is obscure, but are all a natural consequence of the conjectured eleven dimensional E_{11} symmetry. In the next section we will consider the membrane multiplets in detail, and in section 2.6 we will find the remaining charge multiplets shown in table 2.

2.5 The membrane charge multiplet

Until this point we have focussed on reproducing the charge multiplets previously found from U-duality. The membrane charge multiplet has not been given in the literature previously. Here we treat the membrane charge in the same way as the particle and string charges and we apply our method to find the membrane multiplet in three, four, five, six, seven and eight dimensions. For a membrane charge we take $p = 2$ in equation (2.18) and look for solutions when $l = k = 0$. Using equation (2.16) we look for roots of the form $(1^{(D-1)}, 2, 3, m_{D+1}, \dots m_{11})$ in the tables of E_{12} roots showing the l_1 representation of E_{11} which are listed at length in [16].

Level	Charge	Dimension of SL(8) tensor	Mass
1	Z^{ab}	1	$\frac{R_2R_3}{l_p^3}$
2	$Z^{abj_1\dots j_3}$	56	$\frac{R_2R_3R_9\dots R_{11}}{l_p^6}$
3	$Z^{abj_1\dots j_6}$	28	$\frac{R_2R_3R_6\dots R_{11}}{l_p^9}$
	$Z^{abj_1\dots j_5,k_1}$	420	$\frac{R_2R_3R_7\dots R_{10}R_{11}^2}{l_p^9}$
4	$Z^{abj_1\dots j_6,k_1\dots k_3}$	1344	$\frac{R_2R_3VR_9\dots R_{11}}{R_4R_5l_p^{12}}$
	$Z^{abj_1\dots j_7,k_1k_2}$	216	$\frac{R_2R_3VR_{10}R_{11}}{R_4l_p^{12}}$
	$2Z^{abj_1\dots j_8,k_1}$	2×8	$\frac{R_2R_3VR_{11}}{l_p^9}$
	$2Z^{abj_1\dots j_7,k_1,l_1}$	280	$\frac{R_2R_3VR_{11}^2}{R_4l_p^9}$
5	$Z^{abj_1\dots j_6,k_1\dots k_6}$	336	$\frac{R_2R_3V^2}{(R_4R_5)^2l_p^{15}}$
	$Z^{abj_1\dots j_7,k_1\dots k_5}$	378	$\frac{R_2R_3VR_7\dots R_{11}}{R_4l_p^{15}}$
	$2Z^{abj_1\dots j_8,k_1\dots k_4}$	2×70	$\frac{R_2R_3VR_8\dots R_{11}}{l_p^{15}}$
	$Z^{abj_1\dots j_7,k_1\dots k_4,l_1}$	3584	$\frac{R_2R_3VR_8\dots R_{10}R_{11}^2}{R_4l_p^{15}}$
	$2Z^{abj_1\dots j_8,k_1\dots k_3,l_1}$	2×378	$\frac{R_2R_3VR_9R_{10}R_{11}^2}{l_p^{15}}$
6	$2Z^{abj_1\dots j_7}$	2×8	$\frac{R_2R_3V^2}{R_4l_p^{18}}$
	$Z^{abj_1\dots j_7,k_1\dots k_7,l_1}$	280	$\frac{R_2R_3V^2R_{11}}{R_4R_5l_p^{18}}$
	$Z^{abj_1\dots j_7,k_1\dots k_6,l_1l_2}$	4200	$\frac{R_2R_3V^2R_{10}R_{11}}{R_4^2R_5l_p^{18}}$
	$Z^{abj_1\dots j_8,k_1\dots k_4,l_1\dots l_3}$	2352	$\frac{R_2R_3VR_8(R_9\dots R_{11})^2}{l_p^{18}}$
	$Z^{abj_1\dots j_8,k_1\dots k_5,l_1l_2}$	1344	$\frac{R_2R_3V^2R_{10}R_{11}}{R_4R_5R_6l_p^{18}}$
	$4Z^{abj_1\dots j_8,k_1\dots k_6,l_1}$	4×216	$\frac{R_2R_3V^2R_{11}}{R_4R_5l_p^{18}}$
	$Z^{abj_1\dots j_8,k_1\dots k_5,l_1,m_1}$	1800	$\frac{R_2R_3V^2R_{11}^2}{R_4R_5R_6l_p^{18}}$

Table 15 — Continued on next page

Table 15 — Continued from previous page

Level	Charge	Dimension of SL(8) tensor	Mass
7	$Z^{abj_1...j_7,k_1...k_7,l_1...l_4}$	2100	$\frac{R_2R_3V^2R_8...R_{11}}{R_4^2l_p^{21}}$
	$3Z^{abj_1...j_8,k_1...k_7,l_1...l_3}$	3×420	$\frac{R_2R_3V^2R_9R_{10}R_{11}}{R_4l_p^{21}}$
	$3Z^{abj_1...j_8,k_1...k_8,l_1l_2}$	3×28	$\frac{R_2R_3V^2R_{10}R_{11}}{l_p^{21}}$
	$2Z^{abj_1...j_8,k_1...k_6,l_1...l_4}$	2×1512	$\frac{R_2R_3V^2R_8R_9R_{10}R_{11}}{R_4R_5l_p^{21}}$
	$2Z^{abj_1...j_8,k_1...k_7,l_1...l_2,m_1}$	2×1280	$\frac{R_2R_3V^2R_{10}R_{11}^2}{R_4l_p^{21}}$
	$3Z^{abj_1...j_8,k_1...k_8,l_1,m_1}$	3×36	$\frac{R_2R_3V^2R_{11}^2}{l_p^{21}}$
	$Z^{abj_1...j_8,k_1...k_6,l_1...l_3,m_1}$	8820	$\frac{R_2R_3V^2R_9R_{10}R_{11}^2}{R_4R_5l_p^{21}}$
8	$Z^{abj_1...j_7,k_1...k_7,l_1...l_7}$	120	$\frac{R_2R_3V^3}{R_4^3l_p^{24}}$
	$3Z^{abj_1...j_8,k_1...k_7,l_1...l_6}$	3×168	$\frac{R_2R_3V^3R_{11}}{R_4^2R_5l_p^{24}}$
	$3Z^{abj_1...j_8,k_1...k_8,l_1...l_5}$	3×56	$\frac{R_2R_3V^3}{R_4R_5R_6l_p^{24}}$
	$2Z^{abj_1...j_8,k_1...k_7,l_1...l_5,m_1}$	2×2800	$\frac{R_2R_3V^3R_{10}R_{11}}{R_4^2R_5R_6l_p^{24}}$
	$Z^{abj_1...j_8,k_1...k_8,l_1...l_3,m_1m_2}$	1008	$\frac{R_2R_3V^2R_9R_{10}R_{11}^3}{l_p^{24}}$
	$4Z^{abj_1...j_8,k_1...k_8,l_1...l_4,m_1}$	4×504	$\frac{R_2R_3V^2R_8R_9R_{10}R_{11}^2}{l_p^{24}}$
	$4Z^{abj_1...j_8,k_1...k_6,l_1...l_6,m_1}$	2520	$\frac{R_2R_3V^3R_{11}}{R_4R_5l_p^{24}}$
	$Z^{abj_1...j_8,k_1...k_7,l_1...l_4,m_1m_2}$	10584	$\frac{R_2R_3V^3R_{10}R_{11}}{R_4^2R_5R_6R_7l_p^{24}}$
	$Z^{abj_1...j_8,k_1...k_8,l_1...l_3,m_1,n_1}$	1512	$\frac{R_2R_3V^3R_{10}R_{11}}{R_4^2R_5R_6R_7l_p^{24}}$
9	$Z^{abj_1...j_8,k_1...k_8,l_1...l_8,m_1...m_8}$	1	$\frac{R_2R_3V^3}{l_p^{27}}$
	$2Z^{abj_1...j_8,k_1...k_7,l_1...l_7,m_1m_2}$	2×945	$\frac{R_2R_3V^3R_9R_{10}R_{11}}{R_4^2R_5l_p^{27}}$
	$5Z^{abj_1...j_8,k_1...k_8,l_1...l_7,m_1}$	5×63	$\frac{R_2R_3V^3R_{11}}{R_4l_p^{27}}$
	$Z^{abj_1...j_8,k_1...k_8,l_1...l_5,m_1...m_3}$	2352	$\frac{R_2R_3V^3R_9R_{10}R_{11}}{R_4R_5R_6l_p^{27}}$
	$4Z^{abj_1...j_8,k_1...k_8,l_1...l_6,m_1m_2}$	4×720	$\frac{R_2R_3V^3R_{10}R_{11}}{R_4R_5l_p^{27}}$
	$Z^{abj_1...j_8,k_1...k_7,l_1...l_6,m_1...m_3}$	7680	$\frac{R_2R_3V^3R_9R_{10}R_{11}}{R_4^2R_5l_p^{27}}$
	$2Z^{abj_1...j_8,k_1...k_7,l_1...l_7,m_1m_2}$	2×945	$\frac{R_2R_3V^3R_{10}R_{11}}{R_4^2l_p^{27}}$
	$Z^{abj_1...j_8,k_1...k_8,l_1...l_4,m_1...m_4}$	1764	$\frac{R_2R_3V^4}{(R_4...R_7)^2l_p^{27}}$
	$Z^{abj_1...j_8,k_1...k_8,l_1...l_6,m_1,n_1}$	7680	$\frac{R_2R_3V^3R_{10}R_{11}^2}{R_4...R_6l_p^{27}}$

Table 15: The membrane charge multiplet in $D=3$/The **147250** of E_8.

The membrane charge multiplet is a representation of E_{11-D} with highest weight λ_{11-D}, in $D=3$ it is the **147250** of E_8, in $D=4$ it is the **912** of E_7 and so on. In $D=3$ the roots in the l_1 representation having coefficients of the form $(1^{(D-1)},2,3,m_{D+1},\dots m_{11})$ identify the $\mathbf{147250}\oplus\mathbf{30380}\oplus\mathbf{3875}\oplus\mathbf{248}\oplus\mathbf{1}$ of E_8 from which we can identify the root vectors of the **147250** and we list the charges in table 15, where the charges at levels (m_{11})

Level	Charge	Dimension of SL(7) tensor	Mass
1	Z^{ab}	1	$\frac{R_3R_4}{l_p^3}$
2	$Z^{ab,j_1\ldots j_3}$	35	$\frac{R_3R_4R_9\ldots R_{11}}{l_p^6}$
3	$Z^{ab,j_1\ldots j_6}$	7	$\frac{R_3R_4V}{R_5l_p^9}$
	$Z^{ab,j_1\ldots j_5,k}$	140	$\frac{R_3R_4VR_{11}}{R_5R_6l_p^9}$
4	$Z^{ab,j_1\ldots j_6,k_1\ldots k_3}$	224	$\frac{R_3R_4V^2R_9R_{10}R_{11}}{R_5^2R_6R_7R_8l_p^{12}}$
	$Z^{ab,j_1\ldots j_7,k_1k_2}$	21	$\frac{R_3R_4VR_{11}^2}{l_p^{12}}$
	Z^{ab,k_1,l_1}	28	$\frac{R_3R_4VR_{11}^2}{l_p^{12}}$
5	$Z^{ab,j_1\ldots j_6,k_1\ldots k_6}$	28	$\frac{R_3R_4V^2}{R_5^2l_p^{15}}$
	$Z^{ab,j_1\ldots j_7,k_1\ldots k_5}$	21	$\frac{R_3R_4V^2}{R_5R_6l_p^{15}}$
	$Z^{ab,j_1\ldots j_7,k_1\ldots k_4,l_1}$	224	$\frac{R_3R_4V^2R_{11}}{R_5R_6R_7l_p^{15}}$
6	$Z^{ab,j_1\ldots j_7,k_1\ldots k_6,l_1l_2}$	140	$\frac{R_3R_4V^2R_{11}}{R_5l_p^{18}}$
	$Z^{ab,j_1\ldots j_7,k_1\ldots k_7,l_1}$	7	$\frac{R_3R_4V^2R_{11}}{R_5l_p^{18}}$
7	$Z^{ab,j_1\ldots j_7,k_1\ldots k_7,l_1\ldots l_4}$	35	$\frac{R_3R_4V^3}{R_5R_6R_7l_p^{21}}$
8	$Z^{ab,j_1\ldots j_7,k_1\ldots k_7,l_1\ldots l_7}$	1	$\frac{R_3R_4V^3}{l_p^{24}}$

Table 16: The membrane charge multiplet in $D = 4$/The **912** of E_7.

Level	Charge	Dimension of SL(5) tensor	Mass
1	Z^{ab}	1	$\frac{R_4R_5}{l_p^3}$
2	$Z^{ab,j_1\ldots j_3}$	20	$\frac{R_4R_5R_9\ldots R_{11}}{l_p^6}$
3	$Z^{ab,j_1\ldots j_5,k}$	36	$\frac{R_4R_5VR_{11}}{R_6l_p^9}$
4	$Z^{ab,j_1\ldots j_3}$	20	$\frac{R_4R_5V(R_9\ldots R_{11})}{l_p^{12}}$
5	Z^{ab}	1	$\frac{R_4R_5V^2}{l_p^{15}}$

Table 17: The membrane charge multiplet in $D = 5$/The **78** of E_6.

10 to 17 which are the duals of those at levels 8 to 1 are not shown.

The associated invariant mass squared for the membrane charge multiplet in table 15 is

$$\mathcal{M}^2_{\mathbf{147250}} \sim \frac{(R_aR_b)^2V^6}{l_p^{54}} \tag{2.34}$$

In $D = 4$ we look for roots of the form $(1^3, 2, 3, m_5, \ldots m_{11})$ and find the set of charges in table 16, which form the **912** of E_7. The associated invariant mass squared for the membrane charge multiplet in table 16 is

$$\mathcal{M}^2_{\mathbf{912}} \sim \frac{(R_aR_b)^2V^3}{l_p^{27}} \tag{2.35}$$

In $D = 5$ we find the **78** of E_6 as shown in table 17. The associated invariant mass squared

D	Z^{ab}	Z^{abj}	$Z^{abj_1j_2}$	$Z^{abj_1\cdots j_3}$	$Z^{abj_1\cdots j_4}$	$Z^{abj_1\cdots j_5}$	$Z^{abj_1\cdots j_6}$	$Z^{abj_1\cdots j_7}$
3	$1^{(1)}$			$56^{(2)}$			$420^{(3)}$	
		$1856^{(4)}$			$5152^{(5)}$			$11696^{(6)}$
			$16408^{(7)}$			$23472^{(8)}$		
	$29128^{(9)}$			$23472^{(10)}$			$16408^{(11)}$	
		$11696^{(12)}$			$5152^{(13)}$			$1856^{(14)}$
			$420^{(15)}$			$56^{(16)}$		
	$1^{(17)}$							
4	$1^{(1)}$			$35^{(2)}$			$147^{(3)}$	-
			$273^{(4)}$			$273^{(5)}$		-
		$147^{(6)}$			$35^{(7)}$			-
	$1^{(8)}$							
5	$1^{(1)}$			$20^{(2)}$			-	-
	$36^{(3)}$			$20^{(4)}$			-	-
	$1^{(5)}$							
6	$1^{(1)}$			$10^{(2)}$		-	-	-
		$5^{(3)}$				-	-	-
7	$1^{(1)}$			$4^{(2)}$	-	-	-	-
8	$1^{(1)}$			-	-	-	-	-
	$1^{(2)}$			-	-	-	-	-

Table 18: The membrane charge multiplets in $3 \leq D \leq 8$.

for the membrane charge multiplet in table 17 is

$$\mathcal{M}^2_{\mathbf{78}} \sim \frac{(R_a R_b)^2 V^2}{l_p^{18}} \tag{2.36}$$

We can continue this process and find the $\mathbf{1} \oplus \mathbf{10} \oplus \mathbf{5} = \mathbf{16}$ of SO(5,5) in $D = 6$, the $\mathbf{1} \oplus \mathbf{4} = \mathbf{5}$ of SL(5) in $D = 7$ and the $\mathbf{1} \oplus \mathbf{1}$ =(1.22) of SL(3) $\otimes$ SL(2) in $D = 8$. We summarise the appearance of the membrane multiplet charges from the l_1 representation of E_{11} in table 18.

2.6 p-brane charges associated to general weights of E_{11-D}

In the previous sections we considered representations of $A_{D-1} \otimes E_{11-D}$ in the l_1 representation of E_{11}. The representations of A_{D-1} had a single set of antisymmetric indices and corresponded to p-brane charges, the representations of E_{11-D} we considered all had a highest weight which was a single fundamental weight of E_{11-D}. Now we will continue to consider p-brane charges in the A_{D-1} sub-algebra but we will generalise our considerations of E_{11-D} to include those representations whose highest weight is a sum of fundamental weights. As we shall see, representations of E_{11-D}, with highest weight more general than a single fundamental weight, provide a straightforward constraint for the root coefficient m_D. By including the most general highest weight representations of E_{11-D} we will complete table 2.

So far we have not considered the significance of blocks of $(11 - D)$ indices appearing in the index structure of the generators in the E_{11-D} sub-algebra. Previously we have restricted ourselves to the constraint that $l = 0$ in equation (2.21), implying that we have ignored representations of E_{11-D} which are identical up to the $\epsilon^{j_1\cdots j_{11-D}}$ tensor. However the appearance of a block of $(11-D)$ indices may be acted upon by the generators of the A_{10} sub-algebra of E_{11} so that a set of trivial internal indices may acquire spacetime indices and the remaining internal indices form part of a non-trivial representation of E_{11-D}. Let us consider a generic highest weight representation of E_{11-D} labelled by $[p_1, p_2 \dots p_{11-D}]$. Putting this into equation (2.9), where instead of λ_i we now have $\sum p_i\lambda_i$, we find, after making use of equation (2.10),

$$\begin{aligned} -n_j &= \sum p_i < \lambda_i, \lambda_j > - m_D < \lambda_1, \lambda_j > \\ &= \begin{cases} \sum_{i\leq j} ip_i + j\sum_{i\geq j}^{8-D} p_i - m_D + \frac{j}{D-2}\left[\sum_i^{8-D} ip_i + 4p_{9-D} + 2p_{10-D} + 3p_{11-D} - m_D\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \leq 8-D \\ \frac{1}{D-2}\left[4\sum_{i\leq 8-D} ip_i + 2(10-D)p_{9-D} + (10-D)p_{10-D} + 2(8-D)p_{11-D} - 4m_D\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 9-D \\ \frac{1}{D-2}\left[2\sum_{i\leq 8-D} ip_i + (10-D)p_{9-D} + 4p_{10-D} + (8-D)p_{11-D} - 2m_D\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 10-D \\ \frac{1}{D-2}\left[3\sum_{i\leq 8-D} ip_i + 2(8-D)p_{9-D} + (8-D)p_{10-D} + (11-D)p_{11-D} - 3m_D\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 11-D \end{cases} \end{aligned} \tag{2.37}$$

We see that we have a general solution giving positive integer values for n_j when,

$$m_D = \sum_{i=1}^{8-D} ip_i + 4p_{(9-D)} + 2p_{(10-D)} + 3p_{(11-D)} + h(D-2) \tag{2.38}$$

Where h is some positive integer, or zero. For this solution we find the simple root coefficients in E_{11-D} are:

$$n_j = \begin{cases} \sum_{i\geq j}^{8-D}(i-j)p_i + 4p_{9-D} + 2p_{10-D} + 3p_{11-D} + hj + h(D-2) & j \leq 8-D \\ 2p_{9-D} + p_{10-D} + 2p_{11-D} + 4h & j = 9-D \\ p_{9-D} + p_{11-D} + 2h & j = 10-D \\ 2p_{9-D} + p_{10-D} + p_{11-D} + 3h & j = 11-D \end{cases} \tag{2.39}$$

With these coefficients the full E_{12} root is:

$$\begin{aligned} \beta &= e_* + k(e_1 + \cdots + e_{D-p}) + (1+k)(e_{D-p+1} + \cdots + e_D) \\ &\quad + \sum_{n=D+1}^{8-D}\left(h - \sum_{i=n}^{8-D} p_i\right)e_n + he_9 + (h + p_{9-D})e_{10} + (h + p_{9-D} + p_{10-D})e_{11} \end{aligned} \tag{2.40}$$

Where p_i are the Dynkin labels for the E_{11-D} sub-algebra, and p without an index indicates the corresponding p-brane charge multiplet. We can write the parameter, h, as:

$$h = \sum_{i=1}^{8-D} p_i + C \tag{2.41}$$

If we decompose the E_{11-D} representation into SL$(11-D)$ tensor representations the number of SL$(11-D)$ indices appearing on generators at level n_{11-D} in the decomposition, # is given by,

$$\# = 3n_{11-D} - m_D = \sum_{i=1}^{10-D}(11-D-i)p_i + C(11-D) \tag{2.42}$$

So that C controls the appearance of blocks of $(11-D)$ indices. We note that the lower bound on the number of indices, # occurs when n_{11-D} and $p_1 = m_D$, this implies that C is bounded from below by $C \geq -m_D$. Setting $k=0$ we find,

$$\begin{aligned}\beta^2 &= 1 + p + \sum_{i=1}^{8-D} ip_i^2 - 2p_{9-D}^2 - p_{11-D}^2 \\ &\quad +2\sum_{i\leq j} ip_ip_j - 2p_{9-D}p_{10-D} - 4p_{9-D}p_{11-D} - 2p_{10-D}p_{11-D} \\ &\quad -(D-2)h^2 - 2h\sum_{i=1}^{8-D} ip_i - 8hp_{9-D} - 4hp_{10-D} - 6hp_{11-D}\end{aligned} \tag{2.43}$$

Since $\beta^2 = 2, 0, -2, \ldots$ equation (2.43) places constraints on which highest weights of E_{11-D} are present in the l_1 representation. Let us see how useful this equation can be by looking for all the highest weight representations of E_{11-D} appearing in the l_1 representation of E_{11} for the particle, string and membrane charges.

Consider the particle charge multiplet, having $p = 0$, and $m_D = 1$, and we will commence with the D-independent case where $h = 0$. Earlier we found the particle charge multiplet associated to the first fundamental representation of E_{11-D}, whose highest weight is λ_1. From equation (2.38) we see that indeed $p_1 = 1$ is the only possible representation, and from equation (2.43) we find $\beta^2 = 2$ and so it is present in the l_1 representation. However if we now generalise our considerations to include the cases with $h > 0$ then from equation (2.38) we see that in $D = 3$ a new possible particle charge multiplet appears associated to a different highest weight of E_{11-D}. In $D = 3$ we can satisfy $m_D = 1$ with $p_i = 0, h = 1$, and from equation (2.43) we see this corresponds to a root of E_{12} with a squared length of zero. We now identify the complete multiplet associated to this particle charge by looking for roots having the form $(1^4, m_{D+1}, \ldots, m_{11})$. The highest weight root has $\beta^2 = 0$ and under the action of the SL(11) sub-algebra the root length squared may be lowered or held constant, but not raised - due to the Serre relations. Consequently the highest weight representation may be found among the set of roots having the form $(1^4, m_{D+1}, \ldots, m_{11})$ and subject to $\beta^2 \leq 0$. The extra particle multiplet in $D = 3$ is given in table 19. From table 19 associated invariant mass squared for the trivial representation is:

$$\mathcal{M}_{\mathbf{1}}^2 \sim \frac{V^2}{l_p^{18}} \tag{2.44}$$

Let us repeat this process for the string charge multiplets, commencing as before with the case $h = 0$. Putative string charge multiplets have $p = 1$ and $m_D = 2$. From equation (2.38)

E_{12} root	Charge	Dimension of SL(8) tensor	Mass
$(1^5, 2, 3, 4, 5, 3, 1, 3)$	Z	1	$\frac{R_4 \ldots R_{11}}{l_p^9}$

Table 19: A second particle charge multiplet in $D = 3$/The **1** of E_8.

we find three possible highest weight representations:

$$(p_1 = 2), \quad (p_2 = 1), \quad (p_{10-D} = 1) \tag{2.45}$$

However from equation (2.43) (with $h = 0$) we notice that we face the restriction

$$\begin{aligned} 2 \geq 1 + p + \sum_{i=1}^{8-D} i p_i^2 - 2p_{9-D}^2 - p_{11-D}^2 \\ +2 \sum_{i \leq j} i p_i p_j - 2p_{9-D}p_{10-D} - 4p_{9-D}p_{11-D} - 2p_{10-D}p_{11-D} \end{aligned} \tag{2.46}$$

Bearing in mind that for the string $p+1 = 2$ we observe that the remaining positive terms must be balanced or outweighed by the negative terms. That is, any weight with $p_{i \leq 8-D} \neq 0$ must at least be accompanied by non-zero values of p_{9-D} or p_{11-D}. Consequently the putative string charge representations with $(p_1 = 2)$ and $(p_2 = 1)$ do not appear in the l_1 representation. Thus the $(p_{10-D} = 1)$ representation having highest weight λ_{10-D} is the unique representation carrying the string charge in the l_1 representation when $h = 0$. If we now consider solutions when $h \neq 0$ satisfying $m_D = 2 = (p + 1)$ in equation (2.38) we may find the remaining string charge multiplets in $D = 3$. We find,

$$(h = 1, p_1 = 1), \quad (h = 2) \tag{2.47}$$

Where all the remaining p_i are zero. From equation (2.43) we find that $\beta^2 = 0$ for the first case and $\beta^2 = -2$ for the second case. So both of these are highest weights of string charge multiplets in $D = 3$, they are the **248** and the **1** respectively. The string of roots in the representation may be found by searching amongst roots of the form $(1^3, 2, m_{D+1}, \ldots, m_{11})$ in the l_1 representation such that $\beta^2 \leq 0$ in the first case, and $\beta^2 \leq -2$ in the second case. The precise roots forming these multiplets are shown in tables 20 and 21. From table 20 we can identify dual generators in the **248** representation and associate an invariant mass squared to the multiplet:

$$\mathcal{M}_{\mathbf{248}}^2 \sim \frac{R_a^2 V^4}{l_p^{36}} \tag{2.48}$$

The invariant mass squared is:

$$\mathcal{M}_{\mathbf{1}}^2 \sim \frac{R_a^2 V^4}{l_p^{36}} \tag{2.49}$$

We note that the invariant mass squared is the same over all three (the $\mathbf{3875}, \mathbf{248}, \mathbf{1}$) string charge multiplets in three dimensions - the Casimir of the representations of the internal group is a property of the spacetime algebra.

E_{12} root	Charge	Dimension of SL(8) tensor	Mass
$(1^3, 2^2, 3, 4, 5, 6, 4, 2, 3)$	$Z^{aj_1\ldots j_7}$	8	$\frac{R_3 V}{R_4 l_p^9}$
$(1^3, 2, 3, 4, 5, 6, 7, 4, 2, 4)$	$Z^{aj_1\ldots j_8,k_1k_2}$	28	$\frac{R_3 V R_{10} R_{11}}{l_p^{12}}$
$(1^3, 2, 3, 4, 5, 7, 9, 6, 3, 5)$	$Z^{aj_1\ldots j_8,k_1\ldots k_5}$	56	$\frac{R_3 V R_7 \ldots R_{11}}{l_p^{15}}$
$(1^3, 2, 4, 6, 8, 10, 12, 8, 4, 6)$	$Z^{aj_1\ldots j_8,k_1\ldots k_8}$	1	$\frac{R_3 V^2}{l_p^{18}}$
$(1^3, 2, 3, 5, 7, 9, 11, 7, 3, 6)$	$Z^{aj_1\ldots j_8,k_1\ldots k_7,l_1}$	63	$\frac{R_3 V^2 R_{11}}{R_4 l_p^{18}}$
$(1^3, 2, 4, 6, 8, 10, 12, 8, 4, 7)$	$Z^{aj_1\ldots j_8,k_1\ldots k_8,l_1l_2l_3}$	56	$\frac{R_3 V^2 R_9 \ldots R_{11}}{l_p^{21}}$
$(1^3, 2, 4, 6, 9, 12, 15, 10, 5, 8)$	$Z^{aj_1\ldots j_8,k_1\ldots k_8,l_1\ldots l_6}$	28	$\frac{R_3 V^2 R_6 \ldots R_{11}}{l_p^{24}}$
$(1^3, 2, 5, 8, 11, 14, 17, 11, 5, 9)$	$Z^{aj_1\ldots j_8,k_1\ldots k_8,l_1\ldots l_8,m_1}$	8	$\frac{R_3 V^3 R_{11}}{l_p^{27}}$

Table 20: A second string charge multiplet in $D = 3$/The **248** of E_8.

E_{12} root	Charge	Dimension of SL(8) tensor	Mass
$(1^3, 2, 4, 6, 8, 10, 12, 8, 4, 6)$	$Z^{aj_1\ldots j_8,k_1\ldots k_8}$	1	$\frac{R_3 V^2}{l_p^{18}}$

Table 21: A third string charge multiplet in $D = 3$/The **1** of E_8.

We can carry out the same analysis for the possible membrane charge multiplets. With $p = 2$, we look for p_i giving $m_D = 3$ with $h = 0$ in equation (2.38). The possible highest weights have non-zero Dynkin labels:

$$(p_1 = 3), \quad (p_1 = 1, p_2 = 1), \quad (p_3 = 1), \quad (p_1 = 1, p_{10-D} = 1), \quad (p_{11-D} = 1) \qquad (2.50)$$

For $p = 2$ we must find a negative contribution to the squared root length in equation (2.46) coming from the E_{11-D} Dynkin labels, p_i. This means that at least either p_{9-D} or p_{11-D} must be non-zero. Consequently the highest weight representation carrying the membrane charge is unique in the l_1 representation, the **147250**, having $(p_{11-D} = 1)$. When $h > 0$ we find the following highest weights carrying spacetime membrane charges with $\beta^2 = 2, 0, -2, \ldots$:

$$(h = 1, p_2 = 1), \quad (h = 1, p_{10-D} = 1), \quad (h = 2, p_1 = 1), \quad (h = 3) \qquad (2.51)$$

Their highest weights have root lengths such that $\beta^2 \leq 0, -2, -4, -6$ and they correspond to the **30380**, **3875**, **248** and **1** respectively, as indicated in table 2.

This process can be carried out to discover all the charge multiplets from the particle multiplet to the $D - 1$-brane multiplet in D dimensions. The results of doing so are shown in table 2 for three, four, five, six, seven and eight spacetime dimensions.

The reader will notice that in addition to the well-known charge multiplets there are additional representations of E_{11-D} appearing in table 2. For example the particle charge multiplet in three dimensions is the **248** supplemented by a singlet **1**, and the string multiplet is no longer just the **3875** but also the **248** and the **1** of E_8 and so on. The existence of some of these extra multiplets were in fact argued for in [9] see for example

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section 7.2 or appendix C therein. The authors of [9] considered bound states formed from the particle and string charge multiplets, given in the M-theory language, and then looked to see if all the bound states of D-branes in the IIA or IIB language appeared. By identifying the omissions and arguing in favour of completing the spectrum of bound states extra charge multiplets were conjectured. The examples of [9] were in D=5 where a singlet corresponding to a bound state of a $D6$ and a $D0$ brane in the IIA picture was predicted and in D=4 where an additional **56** and two singlets of E_7 were argued for to account for missing bound states of the $D7$ brane, now in the IIB picture. It is straightforward to confirm that the mass of the exotic states in these examples agrees with the masses associated to the matching additional representations shown in table 2. The natural appearance of additional charge multiplets from E_{11}, completing the spectrum of bound states in D=5 and D=4, is an interesting result tied up with the prediction that there exist further exotic states beyond those previously uncovered in the BPS spectrum but revealed fully here.[3]

3. Mass and tension in toroidally compactified backgrounds

The identity between the U-duality symmetries of M-theory and the action of the Weyl group of E_n was achieved in [8, 9] by making use of a formula that gave the tension encoded in a weight vector.[4]

The tension formula was justified empirically but its origin appeared mysterious. The tension formula gave the known tension of p-branes, strings and particles corresponding to a weight vector in a $D+1$ dimensional vector space modulo the addition of the unique vector orthogonal to a D dimensional (sub-)vector space associated with spacetime. The unique orthogonal vector was shown to correspond to Newton's gravitational constant. In this section we give the main result of this paper: an algebraic formula for deriving tensions from the l_1 charge algebra and show that it is consistent with the lower dimensional formula used in [6−10]. We apply the solution to find known brane and string tensions in M-theory, and the IIA and IIB theories, and retrospectively to reproduce the tensions of the brane charge multiplets derived in section 2. Furthermore we are able to interpret the most part of the charge algebra as being associated to KK-brane charges and offer a classification scheme for these exotic charges within the l_1 representation.

3.1 The truncated group element as a vielbein

The group element of $l_1 \otimes_s E_{11}$ at low levels is,

$$g = \exp(x^\mu P_\mu)\exp(h_a{}^b K^a{}_b)\exp\left(\frac{1}{3!}A_{c_1c_2c_3}R^{c_1c_2c_3}\right)\exp\left(\frac{1}{6!}A_{d_1\ldots d_6}R^{d_1\ldots d_6}\right)\ldots \tag{3.1}$$

Where the ellipsis indicates the exponentiation of further generators of both the l_1 representation of E_{11} as well as its adjoint representation. The restriction of the group element

[3] We thank the reviewer for drawing our attention to the comments in this paragraph.

[4] See, for example, equation (3.13) on page 30 of [9]

to the Cartan subalgebra, leaving only the components ${K^a}_a$ of the ${K^a}_b$ generators non-zero, corresponds in the nonlinear realisation to considering a diagonalised vielbein and metric,

$$g_{\mu\mu} = {(e^{2h})_\mu}^a \eta_{aa} \tag{3.2}$$

We recall the success in reconstructing the G^{+++} half-BPS solutions of the eleven dimensional, IIA and IIB supergravity theories [4] as well as the maximally oxidised supergravity theories [13] from the coefficients of the Cartan subalgebra in a truncated version of the group element given by,

$$g_\beta = \exp\left(-\frac{1}{\beta^2}\ln NH\cdot\beta\right)\exp\left((1-N)E_\beta\right) \tag{3.3}$$

This group element encoded half-BPS solutions as a group element of E_{11}. In this group element the part of a general root, β, occurring in the Cartan sub-algebra was singled out using the inner product of the root with the Cartan sub-algebra, $H\cdot\beta$. However in the present paper we work with the l_1 representation of E_{11}, or charge algebra, which is defined in the twelve-dimensional lattice of E_{12}. In this context it is natural to extend the form of the half-BPS group element of E_{11} to E_{12}, where we allow the inner product $H\cdot\beta$ to run over the twelve generators of the Cartan sub-algebra of E_{12}.

It will be fruitful to employ a change of notation at this stage in order to highlight the physical role played by the variables in the group element. There are two natural bases to work in, one uses the diagonalised generators, ${K^a}_a$, and the other uses the generators of the Cartan sub-algebra, H_a. We now make a change of notation and substitute p_a for the field h_a associated to the generators, ${K^a}_a$. When we work in the basis of the Cartan subalgebra, H_a, we will label the fields pre-multiplying the H_a by q_a.

Let us consider a spacetime with a dimension compactified on a circle, the line element for such a direction becomes,

$${(e^{2p_i})_\mu}^i d\xi_i^2 = dx_\mu^2 \tag{3.4}$$

Where ξ_i is a circular coordinate taking values in the range $[0, 2\pi]$. So that if the radius of compactifcation is R_i in the worldvolume, we find by integrating,

$$p_i = \ln\frac{R_i}{l_p} \tag{3.5}$$

Where l_p is the Planck length, and appears so that the parameters p_i are dimensionless. For the non-compact directions we set all the p_i to the same constant [14].

In the setting of the l_1 representation we must also find an interpretation for p_* associated to the additional coordinate. It was shown in [14] that the scaling of the Planck constant under T-duality was predicted by an E_{11} symmetry. In that paper the non-compact p_i were set equal to the same constant C and the effect of the Weyl reflection in the root α_{11} was denoted by a primed index. It was shown that the non-compact parameters encoded the scaling of Minkowski metric of spacetime with respect to the eleven dimensional metric via,

$$C' - C = \ln\frac{l_p}{l'_p} \tag{3.6}$$

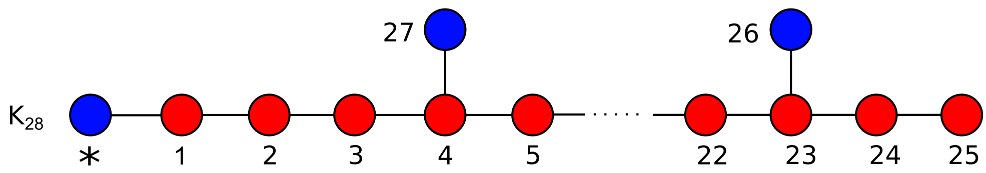

Figure 5: The Dynkin diargam of $K_{28} \equiv D_{24}^{++++}$.

One way to satisfy this scaling is by setting $C = \ln(\frac{1}{l_p}) = p_i$. We propose to treat the extra coordinate in the l_1 representation as such a non-compact parameter and we set:

$$p_* = \ln\left(\frac{1}{l_p}\right) \tag{3.7}$$

And, indeed,

$$p_i = \ln\left(\frac{1}{l_p}\right) \tag{3.8}$$

for all non-compact coordinates. This choice for the non-compact p_i introduces dimensionful parameters into the group theory with the dimension of mass in Planck units. A consequence of p_* being a massive parameter means that the fully-compactified theory will also have a dimensionful parameter.

While this interpretation of the vierbein associated to the additional coordinate is motivated as a particularly simple solution of equation (3.6), it appears appears to be on the correct footing, having a comparable form to that of the compact parameters p_i coordinates, and also being dependent only upon the Planck length. Ultimately we will rest upon the successful reproduction of the brane tensions as shown in the latter sections of this paper and on the correct tensions in the brane charge multiplets of section 2 to justify the choice of p_*.

Upon reduction to the ten dimensional IIA theory we may interpret the radius of the compact direction in terms of the string length,

$$p_{11} = \ln\left(\frac{R_{11}}{l_p}\right) = \ln\left(g_s^{\frac{2}{3}}\right) = \ln\left(\frac{l_p}{l_s}\right)^2 \tag{3.9}$$

Where we have used $R_{11} = g_s l_s$ and $l_p^3 = g_s l_s^3$ which are well-known identifications but which may be independently derived solely from considering the E_{11} algebra [15].

One can make similar speculations for the l_1 extension of the K_{27} algebra related to the twenty-six-dimensional bosonic string, for which the Dynkin diagram is shown below,

In this case we have twenty-six p_i's related to the spacetime coordinates, which we can interpret in a similar way as for E_{11}, i.e. $p_i = \ln\frac{R_i}{l_p}$, and two more, p_{27} and p_*. We would like to make a similar association with energy to the $*$ node, i.e. $p_* = \frac{1}{l_p}$, and we also have the dimensionful constant of string length which we associate with p_{27}, specifically one might take $p_{27} = (\frac{l_s}{l_p})^2$, and see if it leads to sensible and consistent results, but this is beyond the scope of the present work.

3.2 A tension formula

As we have mentioned the l_1 representation is conjectured to be the charge algebra of M-theory. For each brane solution in the E_{11} adjoint algebra there is a corresponding brane charge in the l_1 representation. The l_1 representation is described equivalently by roots in the E_{12} lattice or weights in the E_{11} lattice. For each extremal half-BPS brane the brane charge is equal to the mass. We conjecture the following mass formula for an E_{12} root, β,

$$Z_\beta = \frac{<\beta|\exp\left(q^a\alpha_a\cdot H\right)|\beta>}{<\beta|\beta>} = \exp\left(q^a\alpha_a\cdot\beta\right) \tag{3.10}$$

Where $H_a \equiv \alpha_a \cdot H$ are generators of the E_{12} Cartan subalgebra. It is the expectation value of the half-BPS brane solution group element of equation (3.3) generalised to the E_{12} lattice. Where the Cartan sub-algebra now has twelve generators. We note that the generators which act as raising and lowering operators on $|\beta>$ are projected out as we take the same bra as ket and all that remains of the group element is the Cartan sub-algebra component.

We note here the effect of Weyl reflecting a root β prior to applying the expression in equation (3.10). Recall that the Weyl reflections for a simply laced algebra act on the root lattice as:

$$S_a(\alpha_b) = \alpha_a - 2\frac{<\alpha_a,\alpha_b>}{<\alpha_a,\alpha_a>}\alpha_b = \alpha_a - A_{ab}\alpha_b \tag{3.11}$$

Where A_{ab} is the Cartan matrix. Under this reflection $Z_{S_a(\beta)}$ is:

$$\begin{aligned} Z_{S_a(\beta)} &= \frac{<S_a(\beta)|\exp\left(q^b\alpha_b\cdot H\right)|S_a(\beta)>}{<\beta|\beta>} \\ &= \exp\left(q^b\alpha_b\cdot(S_a\beta)\right) \\ &= Z_\beta\exp\left(-q^bA_{ab}(\alpha_a\cdot\beta)\right) \\ &= S_a(Z_\beta) \end{aligned} \tag{3.12}$$

A Weyl reflection of a root in the E_{12} lattice transforms the radii of the compact physical brane solution and gives a correction to the mass formula for the original brane solution as shown. We will make use of this expression in the IIB scenario to find the change in the mass formula for known solutions which undergo an S-duality transformation.

We may write β in terms of the fundamental weights, l_i, of E_{12},

$$\beta = \sum_{i=*,1}^{11} <\beta,\alpha_i> l_i \equiv \sum \hat{n}_i l_i \tag{3.13}$$

Now, we have,

$$Z_\beta = \exp\left(q^a\alpha_a\cdot\beta\right) \tag{3.14}$$

$$= \exp\left(q^a\hat{n}_a\right) \tag{3.15}$$

The coordinates are now in the Chevalley basis, H_a, whose algebraic interpretation we understand but we would prefer to use a different basis, $K^a{}_a$, whose physical interpretation

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is clear [14]. We change basis to obtain an expression in terms of p^a, the coefficients of ${K^a}_a$, so we may interpret this as a mass formula using equations (3.5) and (3.6). Defining the transformation $\rho : {K^a}_a \to H_a$ and using,

$$\begin{aligned} H_a &= {K^a}_a - {K^{a+1}}_{a+1} \qquad\qquad a = *, 1, 2, \dots 10 \\ H_{11} &= -\frac{1}{3}({K^*}_* + {K^1}_1 + \cdots + {K^8}_8) + \frac{2}{3}({K^9}_9 + {K^{10}}_{10} + {K^{11}}_{11}) \end{aligned} \tag{3.16}$$

Then,

$$-3(\rho^{-1})^T = \begin{pmatrix} -2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 2 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 3 & 3 & 3 & 3 & 3 & 3 & 6 & 6 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 7 & 7 & 7 & 7 & 7 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 9 & 9 & 9 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 7 & 7 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix} \tag{3.17}$$

Now,

$$q^a \alpha_a \cdot H = p^a \cdot \rho^{-1} \cdot \rho \cdot {K^a}_a = p^a \cdot \rho^{-1} \cdot H \tag{3.18}$$

Where we have suppressed the indices on the matrix ρ. For example we may read off,

$$q_* = -\frac{1}{3}(-2p_* + p_1 + p_2 + \cdots p_{11}) \tag{3.19}$$

We now have,

$$\begin{aligned} Z_\beta &= \exp(p^a \cdot \rho^{-1} \cdot \hat{n}_a) \\ &= \exp(\hat{n}^a \cdot (\rho^{-1})^T \cdot p_a) \end{aligned} \tag{3.20}$$

For convenience we make a further basis transformation and express the fundamental weights of E_{12}, l_i, in terms of the 12-dimensional vector space basis, e_i, that we made use of in section 1,

$$\beta = \sum m^i e_i = \sum \hat{n}^j l_j \tag{3.21}$$

We observe that the basis transformation $R : l_i \to e_i$ is $R = (\rho)^T$ and $\hat{n}^i {\rho^{-1}}^T = m^i$, giving a more convenient form for Z_β when substituted in (3.20),

$$Z_\beta = \exp(m \cdot p) \tag{3.22}$$

And making use of equations (3.5) and (3.7), we find a formula that we may interpret as a mass for toroidally compactified dimensions,

$$Z_\beta = \left(\frac{1}{l_p}\right)^{m_*} \prod_{i=1}^{i=11} \left(\frac{R_i}{l_p}\right)^{m_i} \tag{3.23}$$

Where $\beta = \sum_{i=*,1}^{11} m_i e_i$. For the l_1 representation $m_* = 1$. This formula gives the results found in [6−10] directly from the algebra without the need to work modulo the addition of an orthogonal vector corresponding to constants of the theory. In particular see equation (4.28f) on page 54 of the review [9] where the mass of the Kaluza-Klein mode is found modulo the addition of a vector orthogonal to the fundamental weights of the E_{11-D} algebra.

We observe, in passing, that in the review [9] there is an orthogonal vector which in the presentation here is played by the y vector used in section 1 - i.e. it represents the part of the l_1 representation that is orthogonal to the roots of the E_n algebra. In [9] this orthogonal vector encoded Newton's constant. In the present case y too is a function of Newton's constant. The vector $y = e_* - \frac{1}{2}(e_1 + \cdots e_{11})$ has a mass, Z_y where,

$$Z_y = \frac{1}{l_p}\left(\frac{l_p^{11}}{R_1 \dots R_{11}}\right)^{\frac{1}{2}} = \sqrt{G_{11}} = \kappa_{11} \tag{3.24}$$

Where κ_{11} is the gravitational constant in eleven dimensions. Thus the orthogonal vector y encodes the gravitational coupling constant, κ_{11}.

3.3 Brane tensions from weights of the l_1 representation

Now we apply the mass equation (3.23) to low level weights in the l_1 representation which correspond in the various decompositions detailed in section 1 to charges of M-theory, the IIA string theory and the IIB string theory. The process is simplified immensely by deriving the tension in a background where the brane solution is wrapping a torus. That is the charge, which is identified with the mass for the extremal solutions, is contained entirely on the surface of a torus.

3.3.1 Compactifications of M-theory

As described we compactify each p-brane solution on a p-torus in the eleven dimensional background. In this case the tension is derived from the mass by dividing by the p-torus volume. Let us look at specific examples occurring at low levels in the l_1. The relevant decomposition is given explicitly in section 1, and the algebra is split into representations of A_{10} graded by the level m_{11} given in equation (1.15). At level m_{11} we must solve

$$-A + 1 + 11k = 3m_{11} \tag{3.25}$$

With the exception of the translation generator (for which k=0) we will restrict ourselves to the solutions where $k = \sum p_i$, which corresponds to generators with no blocks of eleven antisymmetrised indices (i.e. trivial volume forms ϵ), in which case we may rearrange this formula to get an expression in terms of the number of indices $\# \equiv \sum(11-i)p_i$,

$$\# = 3m_{11} - 1 \tag{3.26}$$

At level $m_{11} = 0$ the corresponding charge has -1 indices, which we may interpret as one contravariant index. The solution has $k = 0$, $p_1 = 1$ and all other $p_i = 0$ and corresponds to the charge P_a. The corresponding root from equation (1.22) and its mass according to equation (3.23) are

$$\beta_{pp} = e_* - e_1 \qquad Z_{\beta_{pp}} = \frac{1}{R_1} \tag{3.27}$$

Where we have singled out R_1 as the radius of a compact direction in which the pp-wave circulates. This is the mass of a KK-mode, or compactified pp-wave solution, in which momentum circulates around the compact direction. The radius R_1 has been singled out in this example, but there is a democracy of spacetime coordinates that can be seen under the Weyl reflections of the gravity line. The simple root we have used here is α_* but under the Weyl reflections of A_{10} may be rotated into $\alpha_* + \alpha_1 + \cdots + \alpha_i$ where $i \leq 10$, which have mass $\frac{1}{R_i}$. In what follows the results will be given modulo the A_{10} Weyl reflections and particular radii will be given in the mass formulae but as in this example there are no inherently special spacetime directions particular to the solution.

At level $m_{11} = 1$ we find a two index charge, associated to the $M2$ brane charge. Specifically we solve equation (1.15),

$$-A + 11k = 2 \tag{3.28}$$

This has a solution with $p_9 = 1$ and all other $p_i = 0$ (implying $k = \sum p_i = 1$), such that by equation (1.24) $\beta^2 = 2$. The corresponding root, from equation (1.22), and its mass according to equation (3.23), are,

$$\beta_{M2} = e_* + e_{10} + e_{11} \qquad Z_{\beta_{M2}} = \frac{R_{10}R_{11}}{l_p^3} \tag{3.29}$$

We calculate the tension by dividing through by the torus volume that the brane is wrapping, for the $M2$ brane wrapped on a 2-torus in the x^{10}, x^{11} directions we divide by $V_2 = (2\pi)^2 R_{10}R_{11}$ giving,

$$T_{M2} \equiv \frac{Z_{\beta_{M2}}}{V_2} = \frac{1}{(2\pi)^2 l_p^3} \tag{3.30}$$

This is the tension [26] of the M2 brane.

At level $m_{11} = 2$ we find the charge corresponding to the $M5$ brane, having $p_6 = 1$ and all other $p_i = 0$ such that $\beta^2 = 2$, giving a root with mass,

$$\beta_{M5} = e_* + e_7 + \cdots + e_{11} \qquad Z_{\beta_{M5}} = \frac{R_7R_8\ldots R_{11}}{l_p^6} \tag{3.31}$$

Dividing through by the surface area of a 5-torus, $V_5 = (2\pi)^5 R_7R_8\ldots R_{11}$, we find the tension [19] of the $M5$ brane,

$$T_{M5} \equiv \frac{Z_{\beta_{M5}}}{V_5} = \frac{1}{(2\pi)^5 l_p^6} \tag{3.32}$$

At level $m_{11} = 3$ we find the charge corresponding to the dual graviton or $KK6$ monopole, having $p_4 = p_7 = 1$ and all other $p_i = 0$ such that $\beta^2 = 2$, giving a root with mass,

$$\beta_{KK6} = e_* + e_5 + \cdots + e_{10} + 2e_{11} \qquad Z_{\beta_{KK6}} = \frac{R_5R_6\ldots R_{10}R_{11}^2}{l_p^9} \tag{3.33}$$

Dividing through by the surface area of a 7-torus, $V_5 = (2\pi)^7 R_5 R_8 \dots R_{11}$, we find the tension,

$$T_{KK6} \equiv \frac{Z_{\beta_{KK6}}}{V_7} = \frac{R_{11}}{(2\pi)^7 l_p^9} \tag{3.34}$$

The tension here is sensible only in the compact setting. Upon decompactifying ($R_{11} \to \infty$) the tension diverges. This is an example of the tension of a Kaluza-Klein brane, the higher-dimensional analogue of the Kaluza-Klein monopole. Let us define a KK-brane to be objects related to p-branes by U-duality transformations whose tension is dependent upon the radii of compactification of the background. The $KK6$ has a Taub-NUT fibration in the R_{11} direction, in our notation. We will find that most tensions arising from the charge algebra will diverge when carried over to the non-compact setting and associated to KK-branes carrying a generalisation of the four-dimensional Taub-NUT charge (for a discussion of the higher-dimensional Taub-NUT charge see [22]).

Let us now turn away from specific cases and look at the mass given by the general E_{12} root in the l_1 representation of E_{11}. By this we mean putting the coefficients of the solution given in equation (1.22) into equation (3.23). We find,

$$Z_\beta = \frac{1}{l_p} \prod_{n=1}^{10} \left(\frac{R_n}{l_p}\right)^{(k-\sum_{i=n}^{10} p_i)} \left(\frac{R_{11}}{l_p}\right)^k \tag{3.35}$$

$$= \frac{R_1^{k-(p_1+\dots+p_{10})} R_2^{k-(p_2+\dots+p_{10})} \dots R_{10}^{k-p_{10}} R_{11}^k}{l_p^{3m_{11}}} \tag{3.36}$$

To find the tension we now divide through by the volume of the relevant compact torus. And now we hit a snag. In our previous tension calculations we knew the charge and hence the brane solution we were considering ab initio. It was therefore clear that for a p-brane solution we could wrap the charge on a p-torus and then divide through by the volume of the p-torus to find the tension. In our approach to a generalised tension formula we do not know the particular solution and neither do we know relevant p-torus volume to divide the mass formula by. In the general case we expect a monomial in the radii to remain even after we have divided the mass formula by the appropriate p-torus volume. We have already seen an example of this in the case of the KK6 or dual graviton tension, where a single power of R_{11} remained in the expression for the tension.

We shall therefore insert a parameter, Ω, that will keep track of all the remaining radii after the appropriate p-torus volume has been divided out. This parameter is introduced to keep our expressions compact and in any particular solution it will be straightforward to give Ω explicitly. Our next step is to divide the mass formula by the part of its numerator which corresponds to the unknown p-torus volume with each radius dressed up with 2π. That is we divide by a volume $V_p \sim (2\pi R)^p$, with one $2\pi R_i$ for each radius that occurs, where,

$$\begin{aligned} V_p\Omega &\equiv (2\pi R_1)^{k-(p_1+\dots+p_{10})}(2\pi R_2)^{k-(p_2+\dots+p_{10})} \dots (2\pi R_{10})^{k-p_{10}}(2\pi R_{11})^k \\ &= (2\pi)^{3m_{11}-1} R_1^{k-(p_1+\dots+p_{10})} R_2^{k-(p_2+\dots+p_{10})} \dots R_{10}^{k-p_{10}} R_{11}^k \end{aligned} \tag{3.37}$$

Returning to our general formula and dividing by the volume V_p of the relevant p-torus we find the tension for all solutions of the l_1 representation is:

$$T_\beta = \frac{\Omega}{(2\pi)^{3m_{11}-1} l_p^{3m_{11}}} = \frac{\Omega}{(2\pi)^{\#} l_p^{\#+1}} \tag{3.38}$$

Where we have used $\# \equiv -A + 11k$, which for the special class of roots with $k = \sum_i p_i$ is the number of indices on the charge.

We observe now that Ω does record a property of the solution, for whenever Ω is not a constant the solution has a divergent tension in the non-compact setting. In this case the tension has a form associated to KKp-branes - objects included by duality arguments into brane charge multiplets in the eleven dimensional theory but really associated to KK-waves and winding modes in lower dimensions. For example in the case of the $KK6$ brane tension considered earlier $\Omega = 2\pi R_{11}$. Indeed in [6, 8, 17, 21] non-perturbative sets of higher-dimensional Kaluza-Klein branes have been found and their masses given for M-theory, IIA and IIB theories. The KK-branes of string theory found in [21] constitute a class of solutions derived from the D7 brane and have masses proportional to $\frac{1}{g_s^3}$. Indeed objects whose mass, Z, is proportional to $\frac{1}{g_s^n}$ where $n \geq 3$ have a non-vanishing gravitational field strength,$F \propto GZ \propto g_s^2 Z$ in the weak coupling limit ($g_s \to 0$). States with these masses do not therefore admit an asymptotically flat spacetime [6]. The weak coupling limit is not sensible for these states, and their presence in M-theory remains an unsolved puzzle. We will identify in the l_1 representation the states of this sort including those discussed in [21] and for the M-theory, IIA and IIB theories we will find highly non-perturbative masses proportional to arbitrary positive powers of $\frac{1}{l_p}$ for M-theory and $\frac{1}{g_s}$ for the string theories.

In the l_1 representation of E_{11} we find the following roots in the E_{12} root lattice with associated mass as shown,

$$\begin{aligned} \beta_{WM_7} &= e_* + 2(e_4 + \cdots + e_{10}) + 3e_{11} & Z_{WM_7} &= \frac{(R_4 \ldots R_{10})^2 R_{11}^3}{l_p^{18}} \\ \beta_{M2_6} &= e_* + (e_4 + e_5) + 2(e_6 + \cdots e_{11}) & Z_{M2_6} &= \frac{R_4 R_5 (R_6 \ldots R_{11})^2}{l_p^{15}} \\ \beta_{M5_3} &= e_* + (e_4 + \cdots + e_8) + 2(e_9 + \cdots e_{11}) & Z_{M5_3} &= \frac{R_4 \ldots R_8 (R_9 \ldots R_{11})^2}{l_p^{12}} \end{aligned} \tag{3.39}$$

These roots have masses which agree with deformations of the fundamental states of M-theory that were found in [21]. These states were argued to exist in the eleven dimensional theory in order to explain the origin of ten-dimensional string theory KK-branes whose existence is inferred from U-duality transformations. Following the notation of [21] the additional M-theory solutions are labelled WM_7, $M2_6$ and $M5_3$. It was recognised [21] that these charges appear in the superalgebra but the attempt to include them in the expansion of the anticommutator, $\{Q, \bar{Q}\}$, were artificial whereas the supercharges arise naturally in the l_1 representation of E_{11}.

We emphasise that these are a small set of all the KK-branes of M-theory the full set of which is contained in the l_1 representation of E_{11}. Indeed it seems the role of

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Level (m_{11})	Mp_i	Mass	E_{12} Root (e_i basis)	Root length squared
3	$M6_1$	$\frac{R_5R_6R_7R_8R_9R_{10}R_{11}^2}{l_p^9}$	$(1,0,0,0,0,1,1,1,1,1,1,2)$	2
4	$M5_3$	$\frac{R_4R_5R_6R_7R_8R_9^2R_{10}^2R_{11}^2}{l_p^{12}}$	$(1,0,0,0,1,1,1,1,1,2,2,2)$	2
4	$M7_2$	$\frac{R_3R_4R_5R_6R_7R_8R_9R_{10}^2R_{11}^2}{l_p^{12}}$	$(1,0,0,1,1,1,1,1,1,1,2,2)$	0
4	$M9_1$	$\frac{R_2R_3R_4R_5R_6R_7R_8R_9R_{10}R_{11}^2}{l_p^{12}}$	$(1,0,1,1,1,1,1,1,1,1,1,2)$	-2
5	$M8_3$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9^2R_{10}^2R_{11}^2}{l_p^{15}}$	$(1,1,1,1,1,1,1,1,1,2,2,2)$	-4
5	$M2_6$	$\frac{R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{15}}$	$(1,0,0,0,1,1,2,2,2,2,2,2)$	2
5	$M4_5$	$\frac{R_3R_4R_5R_6R_7^2R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{15}}$	$(1,0,0,1,1,1,1,2,2,2,2,2)$	0
5	$M6_3$	$\frac{R_2R_3R_4R_5R_6R_7R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{15}}$	$(1,0,1,1,1,1,1,1,2,2,2,2)$	-2
6	$M5_6$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{18}}$	$(1,1,1,1,1,1,2,2,2,2,2,2)$	-6
6	$M1_8$	$\frac{R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{18}}$	$(1,0,0,1,2,2,2,2,2,2,2,2)$	-2
6	$M3_7$	$\frac{R_2R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{18}}$	$(1,0,1,1,1,2,2,2,2,2,2,2)$	-4
7	$M2_9$	$\frac{R_1R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{21}}$	$(1,1,1,2,2,2,2,2,2,2,2,2)$	-10
7	$M0_{10}$	$\frac{R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2R_{11}^2}{l_p^{21}}$	$(1,0,2,2,2,2,2,2,2,2,2,2)$	-8

Table 22: The full set of M-theory Mp_i branes from the l_1 representation of E_{11}.

the variable k used in section 1.1 in the decomposition of the l_1 representation classifies the different types of KK-branes that appear. From equation (1.22), or from the expression for the mass, Z_β, one can see that the dependence on the eleventh radius R_{11} is controlled by k. The KK-branes have masses with powers of the radii greater than one, and since k is also the greatest radial power appearing in the mass formula one could use k to classify finite sets of KK-branes. For example in M-theory one could look for all the KK-branes with masses quadratic in the radii by listing the roots with $k = 2$. From table 25, one finds the following, given explicitly in table 22: $M6_1, M5_3, M7_2, M9_1, M8_3, M2_6, M4_5, M6_4, M5_6, M1_8, M3_7, M2_9$ where the script number refers to the number of linear radii and the subscript to the number of radii which are squared in the mass formula.

3.3.2 IIA supergravity

To study the ten-dimensional IIA theory the l_1 algebra is decomposed into representations of the A_9 subalgebra formed from the roots numbered 1 to 9, in the E_{12} Dynkin diagrams shown in the introduction. The decomposition was carried out in section 1 and a general root given in equation (1.39). We now find the formula for the mass this general root as given by equation (3.23),

$$Z_\beta = \frac{1}{l_p}\prod_{n=1}^{9}\left(\frac{R_n}{l_p}\right)^{(k-\sum_{i=n}^{9}p_i)}\left(\frac{R_{10}}{l_p}\right)^{k}\left(\frac{R_{11}}{l_p}\right)^{(m_{11}-m_{10})} \tag{3.40}$$

$$= \frac{R_1^{k-(p_1+\cdots+p_9)}R_2^{k-(p_2+\cdots+p_9)}\dots R_9^{k-p_9}R_{10}^{k}R_{11}^{m_{11}-m_{10}}}{l_p^{3m_{11}}} \tag{3.41}$$

To find a convenient notation we divide through by the appropriate volume V_p as before to remove the radial dependencies, i.e. we divide by

$$V_p \equiv \frac{(2\pi)^{10k-A}}{\Omega}R_1^{k-(p_1+\cdots+p_9)}R_2^{k-(p_2+\cdots+p_9)}\dots R_9^{k-p_9}R_{10}^{k} \tag{3.42}$$

giving,

$$T_\beta = \frac{\Omega R_{11}^{m_{11}-m_{10}}}{(2\pi)^{10k-A}l_p^{3m_{11}}} \tag{3.43}$$

$$= \frac{\Omega}{(2\pi)^{\#}g_s^{m_{10}}l_s^{\#+1}} \tag{3.44}$$

Where we have made use of the identities $R_{11} = g_s l_s$ and $l_p^3 = g_s l_s^3$. As before we have defined $\# \equiv 10k - A$ and in the case where $k = \sum_i p_i$, $\#$ is the number of indices on the corresponding generator in the algebra. We now may pose the question: which roots in the algebra have tensions with a single string coupling constant in them. Immediately we see we are looking for the roots such that $m_{10} = 1$. Now $\# = 2m_{11} + m_{10} - 1 = 2m_{11}$. Therefore we note that the charges we will find matching this criteria will have an even number of indices if they exist in the algebra. Specifically $m_{10} = 1$ gives $q = -k + 1$ and so $2m_{11} = -A + 10k$. The cases where $k = \sum_i p_i$ and where only a single p_i is nonzero and equal to one we have $k = 1$ and we must solve $2m_{11} = 10 - A \geq 0$ which has solutions for $A = 0, 2, 4, 6, 8$ corresponding to the D10, D8, D6, D4, D2 brane solutions. For each Dp-brane we have $p_{(10-p)} = 1$ with all other $p_i = 0$. Therefore for these cases, we have $\# = p$ where p is even and less than ten, we find,

$$T_{\beta_{Dp}} = \frac{1}{(2\pi)^p g_s l_s^{p+1}} \tag{3.45}$$

Which is the tension formula for the RR D-brane charges of the IIA theory. The D0-brane charge occurs when all $p_i = 0$ so that $\# = 0$, consequently,

$$\beta_{D0} = e_* - e_{11} \qquad T_{\beta_{D0}} = \frac{1}{g_s l_s} \equiv \frac{1}{g_s\sqrt{\alpha'}} \tag{3.46}$$

Note that the result is precisely given by by the tension formula from β_{D0} and not modulo an additional orthogonal vector as was the case for the KK mode of [9]. The fundamental string charge is listed at level $(1, 1)$ in table 27 in the appendix - it's highest weight generator is associated to the root $\beta_{F1} = e_* - e_{11}$ and its tension is:

$$T_{\beta_{F1}} = \frac{1}{2\pi l_s^2} \tag{3.47}$$

One can also identify the tension of the $NS5$ brane, whose charge is associated to the root $\beta_{NS5} = e_* + e_6 + e_7 + e_8 + e_9 + e_{11}$ appearing at level $(3, 1)$ in table A3. The corresponding tension is:

$$T_{\beta_{NS5}} = \frac{1}{(2\pi)^5 g_s^2 l_s^6} \tag{3.48}$$

Amongst the weights in the l_1 representation of E_{11} one can use the tension formula to identify the roots corresponding to the KK-branes of the IIA theory given in [21], namely the $D0_7, D2_5, D4_3$ and $D6_1$ which are all derived from the D7 brane by U-duality transformations. Using table 26 in the appendix we may identify:

$$\begin{aligned}
\beta_{D0_7} &= e_* + 2(e_4 + \cdots + e_{10}) + 3e_{11} & Z_{D0_7} &= \frac{(R_4 \ldots R_{10})^2}{g_s^3 l_s^{15}} \\
\beta_{D2_5} &= e_* + (e_4 + e_5) + 2(e_6 + \cdots e_{11}) & Z_{D2_5} &= \frac{R_4 R_5 (R_6 \ldots R_{10})^2}{g_s^3 l_s^{13}} \\
\beta_{D4_3} &= e_* + (e_4 + \cdots + e_7) + 2(e_8 + \cdots e_{10}) + e_{11} & Z_{D4_3} &= \frac{R_4 \ldots R_7 (R_8 \ldots R_{10})^2}{g_s^3 l_s^{11}} \\
\beta_{D6_1} &= e_* + (e_4 + \cdots + e_9) + 2e_{10} & Z_{D6_1} &= \frac{R_4 \ldots R_9 (R_{10})^2}{g_s^3 l_s^9}
\end{aligned} \tag{3.49}$$

One can use the parameter k to classify the KK-brane solutions; in the IIA decomposition k is the coefficient of e_{10}. The full set of Dp_i branes, where i indicates the number of directions with a Taub-NUT fibration, corresponds to all the weights in the l_1 representation having $k = 2$, shown in table 23.

Level (m_{10}, m_{11})	Dp_i	Mass	E_{12} Root (e_i basis)	Root length squared
$(2,3)$	$D5_1$	$\frac{R_5R_6R_7R_8R_9R_{10}^2}{g_s^2l_s^8}$	$(1,0,0,0,0,1,1,1,1,1,2,1)$	2
$(3,3)$	$D6_1$	$\frac{R_4R_5R_6R_7R_8R_9R_{10}^2}{g_s^3l_s^9}$	$(1,0,0,0,1,1,1,1,1,1,2,0)$	2
$(2,4)$	$D5_2$	$\frac{R_4R_5R_6R_7R_8R_9^2R_{10}^2}{g_s^2l_s^{10}}$	$(1,0,0,0,1,1,1,1,1,2,2,2)$	2
$(2,4)$	$D7_1$	$\frac{R_3R_4R_5R_6R_7R_8R_9R_{10}^2}{g_s^2l_s^{10}}$	$(1,0,0,1,1,1,1,1,1,1,2,2)$	0
$(3,4)$	$D4_3$	$\frac{R_4R_5R_6R_7R_8^2R_9^2R_{10}^2}{g_s^3l_s^{11}}$	$(1,0,0,0,1,1,1,1,2,2,2,1)$	2
$(3,4)$	$D6_2$	$\frac{R_3R_4R_5R_6R_7R_8R_9^2R_{10}^2}{g_s^3l_s^{11}}$	$(1,0,0,1,1,1,1,1,1,2,2,1)$	0
$(3,4)$	$D8_1$	$\frac{R_2R_3R_4R_5R_6R_7R_8R_9R_{10}^2}{g_s^3l_s^{11}}$	$(1,0,1,1,1,1,1,1,1,1,2,1)$	-2
$(4,4)$	$D9_1$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9R_{10}^2}{g_s^4l_s^{12}}$	$(1,1,1,1,1,1,1,1,1,1,2,0)$	-2
$(4,4)$	$D5_3$	$\frac{R_3R_4R_5R_6R_7R_8^2R_9^2R_{10}^2}{g_s^4l_s^{12}}$	$(1,0,0,1,1,1,1,1,2,2,2,0)$	2
$(4,4)$	$D7_2$	$\frac{R_2R_3R_4R_5R_6R_7R_8R_9^2R_{10}^2}{g_s^4l_s^{12}}$	$(1,0,1,1,1,1,1,1,1,2,2,0)$	0
$(5,4)$	$D8_2$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9^2R_{10}^2}{g_s^5l_s^{13}}$	$(1,1,1,1,1,1,1,1,1,2,2,-1)$	2
$(2,5)$	$D9_1$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9R_{10}^2}{g_s^2l_s^{12}}$	$(1,1,1,1,1,1,1,1,1,1,2,3)$	-2
$(2,5)$	$D5_3$	$\frac{R_3R_4R_5R_6R_7R_8^2R_9^2R_{10}^2}{g_s^2l_s^{12}}$	$(1,0,0,1,1,1,1,1,2,2,2,3)$	2

Table 23 — Continued on next page

Table 23 — Continued from previous page

Level (m_{10}, m_{11})	Dp_i	Mass	E_{12} Root (e_i basis)	Root length squared
$(2,5)$	$D7_2$	$\frac{R_2R_3R_4R_5R_6R_7R_8R_9^2R_{10}^2}{g_s^2l_s^{12}}$	$(1,0,1,1,1,1,1,1,1,2,2,3)$	0
$(3,5)$	$D8_2$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9^2R_{10}^2}{g_s^3l_s^{13}}$	$(1,1,1,1,1,1,1,1,1,2,2,2)$	-4
$(3,5)$	$D2_5$	$\frac{R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3l_s^{13}}$	$(1,0,0,0,1,1,2,2,2,2,2,2)$	2
$(3,5)$	$D4_4$	$\frac{R_3R_4R_5R_6R_7^2R_8^2R_9^2R_{10}^2}{g_s^3l_s^{13}}$	$(1,0,0,1,1,1,1,2,2,2,2,2)$	0
$(3,5)$	$D6_3$	$\frac{R_2R_3R_4R_5R_6R_7R_8^2R_9^2R_{10}^2}{g_s^3l_s^{13}}$	$(1,0,1,1,1,1,1,1,2,2,2,2)$	-2
$(4,5)$	$D7_3$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8^2R_9^2R_{10}^2}{g_s^4l_s^{14}}$	$(1,1,1,1,1,1,1,1,2,2,2,1)$	-4
$(4,5)$	$D1_6$	$\frac{R_4R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4l_s^{14}}$	$(1,0,0,0,1,2,2,2,2,2,2,1)$	2
$(4,5)$	$D3_5$	$\frac{R_3R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4l_s^{14}}$	$(1,0,0,1,1,1,2,2,2,2,2,1)$	0
$(4,5)$	$D5_4$	$\frac{R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2R_{10}^2}{g_s^4l_s^{14}}$	$(1,0,1,1,1,1,1,2,2,2,2,1)$	-2
$(5,5)$	$D6_4$	$\frac{R_1R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2R_{10}^2}{g_s^5l_s^{15}}$	$(1,1,1,1,1,1,1,2,2,2,2,0)$	-2
$(5,5)$	$D2_6$	$\frac{R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5l_s^{15}}$	$(1,0,0,1,1,2,2,2,2,2,2,0)$	2
$(5,5)$	$D4_5$	$\frac{R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5l_s^{15}}$	$(1,0,1,1,1,1,2,2,2,2,2,0)$	0
$(6,5)$	$D5_5$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^6l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,2,-1)$	2
$(2,6)$	$D7_3$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8^2R_9^2R_{10}^2}{g_s^2l_s^{14}}$	$(1,1,1,1,1,1,1,1,2,2,2,4)$	0
$(2,6)$	$D5_4$	$\frac{R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2R_{10}^2}{g_s^2l_s^{14}}$	$(1,0,1,1,1,1,1,2,2,2,2,4)$	2
$(3,6)$	$D6_4$	$\frac{R_1R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2R_{10}^2}{g_s^3l_s^{15}}$	$(1,1,1,1,1,1,1,2,2,2,2,3)$	-4
$(3,6)$	$D0_7$	$\frac{R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3l_s^{15}}$	$(1,0,0,0,2,2,2,2,2,2,2,3)$	2
$(3,6)$	$D2_6$	$\frac{R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3l_s^{15}}$	$(1,0,0,1,1,2,2,2,2,2,2,3)$	0
$(3,6)$	$D4_5$	$\frac{R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3l_s^{15}}$	$(1,0,1,1,1,1,2,2,2,2,2,3)$	-2
$(4,6)$	$D4_6$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,2,2)$	-6
$(4,6)$	$D1_7$	$\frac{R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4l_s^{16}}$	$(1,0,0,1,2,2,2,2,2,2,2,2)$	-2
$(4,6)$	$D3_6$	$\frac{R_2R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4l_s^{16}}$	$(1,0,1,1,1,2,2,2,2,2,2,2)$	-4
$(5,6)$	$D4_6$	$\frac{R_1R_2R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5l_s^{17}}$	$(1,1,1,1,1,2,2,2,2,2,2,1)$	-6
$(5,6)$	$D0_8$	$\frac{R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5l_s^{17}}$	$(1,0,0,2,2,2,2,2,2,2,2,1)$	-2
$(5,6)$	$D2_7$	$\frac{R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5l_s^{17}}$	$(1,0,1,1,2,2,2,2,2,2,2,1)$	-4
$(6,6)$	$D3_7$	$\frac{R_1R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^6l_s^{18}}$	$(1,1,1,1,2,2,2,2,2,2,2,0)$	-4
$(6,6)$	$D1_8$	$\frac{R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^6l_s^{18}}$	$(1,0,1,2,2,2,2,2,2,2,2,0)$	-2
$(7,6)$	$D2_8$	$\frac{R_1R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^7l_s^{19}}$	$(1,1,1,2,2,2,2,2,2,2,2,-1)$	0
$(7,6)$	$D0_9$	$\frac{R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^7l_s^{19}}$	$(1,0,2,2,2,2,2,2,2,2,2,-1)$	2

Table 23 — Continued on next page

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Table 23 — Continued from previous page

Level (m_{10}, m_{11})	Dp_i	Mass	E_{12} Root (e_i basis)	Root length squared
$(2,7)$	$D5_5$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^2 l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,2,5)$	2
$(3,7)$	$D4_6$	$\frac{R_1R_2R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3 l_s^{17}}$	$(1,1,1,1,1,2,2,2,2,2,2,4)$	-4
$(3,7)$	$D0_8$	$\frac{R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3 l_s^{17}}$	$(1,0,0,2,2,2,2,2,2,2,2,4)$	0
$(3,7)$	$D2_7$	$\frac{R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3 l_s^{17}}$	$(1,0,1,1,2,2,2,2,2,2,2,4)$	-2
$(4,7)$	$D3_7$	$\frac{R_1R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4 l_s^{18}}$	$(1,1,1,1,2,2,2,2,2,2,2,3)$	-8
$(4,7)$	$D1_8$	$\frac{R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4 l_s^{18}}$	$(1,0,1,2,2,2,2,2,2,2,2,3)$	-6
$(5,7)$	$D2_8$	$\frac{R_1R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5 l_s^{19}}$	$(1,1,1,2,2,2,2,2,2,2,2,2)$	-10
$(5,7)$	$D0_9$	$\frac{R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5 l_s^{19}}$	$(1,0,2,2,2,2,2,2,2,2,2,2)$	-8
$(6,7)$	$D1_9$	$\frac{R_1R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^6 l_s^{20}}$	$(1,1,2,2,2,2,2,2,2,2,2,1)$	-10
$(7,7)$	$D0_{10}$	$\frac{R_1^2R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^7 l_s^{21}}$	$(1,2,2,2,2,2,2,2,2,2,2,0)$	-8
$(3,8)$	$D2_8$	$\frac{R_1R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3 l_s^{19}}$	$(1,1,1,2,2,2,2,2,2,2,2,5)$	-4
$(3,8)$	$D0_9$	$\frac{R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3 l_s^{19}}$	$(1,0,2,2,2,2,2,2,2,2,2,5)$	-2
$(4,8)$	$D1_9$	$\frac{R_1R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^4 l_s^{20}}$	$(1,1,2,2,2,2,2,2,2,2,2,4)$	-10
$(5,8)$	$D0_{10}$	$\frac{R_1^2R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^5 l_s^{21}}$	$(1,2,2,2,2,2,2,2,2,2,2,3)$	-14
$(3,9)$	$D0_{10}$	$\frac{R_1^2R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2R_{10}^2}{g_s^3 l_s^{21}}$	$(1,2,2,2,2,2,2,2,2,2,2,6)$	-4

Table 23: The full set of IIA Dp_i branes from the l_1 representation of E_{11}.

3.3.3 IIB supergravity

We repeat the process applied to the IIA case but making use of the IIB variables. The decomposition was carried out in section 1 and a general root given in equation (1.57) which we use with equation (3.6) to find the mass,

$$Z_\beta = \frac{1}{l_p}\prod_{n=1}^{9}\left(\frac{R_n}{l_p}\right)^{(k-\sum_{i=n}^{8,11}p_i)}\left(\frac{R_{10}}{l_p}\right)^{l-k}\left(\frac{R_{11}}{l_p}\right)^{(m_9-k-l)} \tag{3.50}$$

$$= \frac{R_1^{k-(p_1+\cdots+p_8+p_{11})}R_2^{k-(p_2+\cdots+p_8+p_{11})}\ldots R_9^{k-p_{11}}R_{10}^{l-k}R_{11}^{m_9-k-l}}{l_p^{3m_{11}}} \tag{3.51}$$

To find a convenient notation for the tension we divide through by the volume to remove the radial dependencies, i.e. we divide by

$$V_p \equiv \frac{(2\pi)^{(10k-A)}}{\Omega}R_1^{k-(p_1+\cdots+p_8+p_{11})}R_2^{k-(p_2+\cdots+p_8+p_{11})}\ldots R_9^{k-p_{11}}\hat{R}_{10}^k \tag{3.52}$$

This contains the volume of the compact spacetime torus as a factor and as before we record remaining powers of $2\pi R$ in a factor, Ω. We have used the notation $\hat{R}_{10} \equiv \frac{l_p^3}{R_{10}R_{11}}$ [15]

so that,

$$T_\beta = \frac{\Omega}{(2\pi)^{2m_9-1}\hat{g}_s^l l_s^{2m_9}} \tag{3.53}$$

Where we have used the IIB parameters $l_s^2 = \frac{l_p^3}{R_{11}}$ and $\hat{g}_s = \frac{R_{11}}{R_{10}}$. If, as before, we define $\# \equiv 10k - A = 2m_9 - 1$ then we have,

$$T_\beta = \frac{\Omega}{(2\pi)^{\#}\hat{g}_s^l l_s^{\#+1}} \tag{3.54}$$

As before, in the case where $k = \sum_i p_i$, $\#$ is the number of indices on the corresponding generator in the algebra. We now may single out the tensions with a single $\hat{g}^s$ appearing. In this case we are looking for roots with $l = m_{10} = 1$. Now $\# = 2m_9 - 1$. Therefore we note that the charges we will find matching this criteria will have an odd number of indices if they exist in the algebra. Specifically where $k = \sum_i p_i$ and where only a single p_i is nonzero and equal to one we have $k = 1$ and we must solve $2m_9 = 11 - A \geq 0$ which has solutions for $A = 1, 3, 5, 7, 9$ corresponding to the D9, D7, D5, D3, D1 brane solutions. Excluding the D1 brane we have for each Dp-brane, $p_{(10-p)} = 1$ with all other $p_i = 0$. For the D1 brane we have $p_{11} = 1$ and all other $p_i = 0$. Therefore for these cases, we have $\# = p$ where p is odd and less than ten, we find,

$$T_{\beta_{Dp}} = \frac{1}{(2\pi)^p g_s l_s^{p+1}} \tag{3.55}$$

Which is the tension formula for the RR D-brane charges of the IIB theory. Furthermore we may apply the S-duality transformation to the IIB theory which corresponds to the Weyl reflection in the plane perpendicular to the α_{10} root vector:

$$S_{10}\beta = \beta - (2l - m_9)\alpha_{10} \tag{3.56}$$

That is $l \to m_9 - l$, leaving β^2 in equation (1.58) unaltered, as expected, but altering the tension formula to:

$$T'_{\beta_{Dp}} = \frac{1}{(2\pi)^p g_s^{m_9-1} l_s^{p+1}} \tag{3.57}$$

This formula could also be obtained using equation (3.12) for the Weyl reflection of the mass formula. Explicitly we have,

$$Z'_{\beta_{Dp}} \equiv Z_{S_{10}(\beta_{Dp})} = Z_{\beta_{Dp}} \exp\left(-(-q^9 + 2q^{10})(\alpha_{10} \cdot \beta_{D_p})\right) \tag{3.58}$$

Using the parameters of solution given in section 1.3 we have,

$$\alpha_{10} \cdot \beta = -m_9 + 2m_{10} = -q \tag{3.59}$$

We recall that in general q is an integer taking values less than or equal to the coefficient, m_9, but in the particular set of solutions we are considering here $q = m_9 - 2$. Let us translate the $-q^9 + 2q^{10}$ into the coordinates p^a, using the transformation matrix $(\rho^{-1})^T$ from section 3.2 we read off,

$$q^9 = -\frac{1}{3}(4(p_* + p_1 + \cdots p_9) + 7(p_{10} + p_{11})) \tag{3.60}$$

$$q^{10} = -\frac{1}{3}(2(p_* + p_1 + \cdots p_{10}) + 5p_{11})) \tag{3.61}$$

So that,

$$-q^9 + 2q^{10} = p_{10} - p_{11} = \ln\left(\frac{R_{10}}{R_{11}}\right) = \ln\left(\frac{1}{g_s}\right) \tag{3.62}$$

Substituting back into our expression for $Z_{S_{10}(\beta)}$ an dividing by $V_p \sim (2\pi R)^p$ to find the tension we obtain,

$$T_{S_{10}(\beta_{D_p})} = T_{\beta_{D_p}}\left(\frac{1}{g_s}\right)^q = \frac{1}{(2\pi)^p g_s^{1+q} l_s^{p+1}} \tag{3.63}$$

In the case we have been concerned with the parameters take specific values, $l = 1$ and so $q = m_9 - 2$ and the expression is identical to the S-dual tension derived previously.

We may use this form of the tension to write down the tensions of the S-dual states to the Dp-branes of the IIB theory, explicitly the $F1$ string, the $NS5$ brane, the $S7$ brane and the $S9$ brane. Notice that the tension of the D3 brane is mapped to itself by S-duality. Explicitly we find,

$$T_{\beta_{F1}} = \frac{1}{2\pi l_s^2} \tag{3.64}$$

$$T_{\beta_{NS5}} = \frac{1}{(2\pi)^5 g_s^2 l_s^6} \tag{3.65}$$

$$T_{\beta_{S7}} = \frac{1}{(2\pi)^7 g_s^3 l_s^8} \tag{3.66}$$

$$T_{\beta_{S9}} = \frac{1}{(2\pi)^9 g_s^4 l_s^{10}} \tag{3.67}$$

In addition there are two further 9-brane states making up the charge $Z^{a_1\ldots a_9(\alpha\beta\gamma)}$, these are S-dual to each other and have tensions:

$$T_{\beta_9} = \frac{1}{(2\pi)^9 g_s^2 l_s^{10}} \tag{3.68}$$

$$T_{\beta_9} = \frac{1}{(2\pi)^9 g_s^3 l_s^{10}} \tag{3.69}$$

Let us also identify the weights in the l_1 representation corresponding to the KK-branes of the IIB theory given in [21], namely the $D1_6$, $D3_4$ and $D5_2$ which are all derived from the D7 brane by U-duality transformations. From table 27 in the appendix we may read off:

$$\beta_{D1_6} = e_* + 2(e_4 + \cdots + e_{10}) + 3e_{11} \qquad Z_{D0_7} = \frac{(R_4 \ldots R_{10})^2}{g_s^3 l_s^{15}} \tag{3.70}$$

$$\beta_{D3_4} = e_* + (e_4 + e_5) + 2(e_6 + \cdots e_{11}) \qquad Z_{D2_5} = \frac{R_4 R_5 (R_6 \ldots R_{10})^2}{g_s^3 l_s^{13}}$$

$$\beta_{D5_2} = e_* + (e_4 + \cdots + e_7) + 2(e_8 + \cdots e_{10}) + e_{11} \qquad Z_{D4_3} = \frac{R_4 \ldots R_7 (R_8 \ldots R_{10})^2}{g_s^3 l_s^{11}}$$

One can use the parametner k to classify the KK-brane solutions; in the IIB decomposition $m_{10} - k$ is the coefficient of e_{10}. The full set of Dp_i branes, where i indicates the number of directions with a Taub-NUT fibration, corresponds to all the weights in the l_1 representation having $k = 2$ and is shown in table 24.

Level (m_9, m_{10})	Dp_i	Mass	E_{12} Root (e_i basis)	Root length squared
$(4,2)$	$D5_1$	$\frac{R_5R_6R_7R_8R_9\hat{R}_{10}^2}{g_s^2l_s^8}$	$(1,0,0,0,0,1,1,1,1,1,0,0)$	2
$(5,2)$	$D5_2$	$\frac{R_4R_5R_6R_7R_8R_9^2\hat{R}_{10}^2}{g_s^2l_s^{10}}$	$(1,0,0,0,1,1,1,1,1,2,0,1)$	2
$(5,2)$	$D7_1$	$\frac{R_3R_4R_5R_6R_7R_8R_9\hat{R}_{10}^2}{g_s^2l_s^{10}}$	$(1,0,0,1,1,1,1,1,1,1,0,1)$	0
$(5,3)$	$D5_2$	$\frac{R_4R_5R_6R_7R_8R_9^2\hat{R}_{10}^2}{g_s^3l_s^{10}}$	$(1,0,0,0,1,1,1,1,1,2,1,0)$	2
$(5,3)$	$D7_1$	$\frac{R_3R_4R_5R_6R_7R_8R_9\hat{R}_{10}^2}{g_s^3l_s^{10}}$	$(1,0,0,1,1,1,1,1,1,1,1,0)$	0
$(6,2)$	$D9_1$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9\hat{R}_{10}^2}{g_s^2l_s^{12}}$	$(1,1,1,1,1,1,1,1,1,1,0,2)$	-2
$(6,2)$	$D5_3$	$\frac{R_3R_4R_5R_6R_7R_8^2R_9^2\hat{R}_{10}^2}{g_s^2l_s^{12}}$	$(1,0,0,1,1,1,1,1,2,2,0,2)$	2
$(6,2)$	$D7_2$	$\frac{R_2R_3R_4R_5R_6R_7R_8R_9^2\hat{R}_{10}^2}{g_s^2l_s^{12}}$	$(1,0,1,1,1,1,1,1,1,2,0,2)$	0
$(6,3)$	$D9_1$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9\hat{R}_{10}^2}{g_s^3l_s^{12}}$	$(1,1,1,1,1,1,1,1,1,1,1,1)$	-4
$(6,3)$	$D3_4$	$\frac{R_4R_5R_6R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{12}}$	$(1,0,0,0,1,1,1,2,2,2,1,1)$	2
$(6,3)$	$D5_3$	$\frac{R_3R_4R_5R_6R_7R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{12}}$	$(1,0,0,1,1,1,1,1,2,2,1,1)$	0
$(6,3)$	$D7_2$	$\frac{R_2R_3R_4R_5R_6R_7R_8R_9^2\hat{R}_{10}^2}{g_s^3l_s^{12}}$	$(1,0,1,1,1,1,1,1,1,2,1,1)$	-2
$(6,4)$	$D9_1$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8R_9\hat{R}_{10}^2}{g_s^4l_s^{12}}$	$(1,1,1,1,1,1,1,1,1,1,2,0)$	-2
$(6,4)$	$D5_3$	$\frac{R_3R_4R_5R_6R_7R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{12}}$	$(1,0,0,1,1,1,1,1,2,2,2,0)$	2
$(6,4)$	$D7_2$	$\frac{R_2R_3R_4R_5R_6R_7R_8R_9^2\hat{R}_{10}^2}{g_s^4l_s^{12}}$	$(1,0,1,1,1,1,1,1,1,2,2,0)$	0
$(7,2)$	$D7_3$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8^2R_9^2\hat{R}_{10}^2}{g_s^2l_s^{14}}$	$(1,1,1,1,1,1,1,1,2,2,0,3)$	0
$(7,2)$	$D5_4$	$\frac{R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^2l_s^{14}}$	$(1,0,1,1,1,1,1,2,2,2,0,3)$	2
$(7,3)$	$D7_3$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{14}}$	$(1,1,1,1,1,1,1,1,2,2,1,2)$	-4
$(7,3)$	$D1_6$	$\frac{R_4R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{14}}$	$(1,0,0,0,1,2,2,2,2,2,1,2)$	2
$(7,3)$	$D3_5$	$\frac{R_3R_4R_5R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{14}}$	$(1,0,0,1,1,1,2,2,2,2,1,2)$	0
$(7,3)$	$D5_4$	$\frac{R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{14}}$	$(1,0,1,1,1,1,1,2,2,2,1,2)$	-2
$(7,4)$	$D7_3$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{14}}$	$(1,1,1,1,1,1,1,1,2,2,2,1)$	-4
$(7,4)$	$D1_6$	$\frac{R_4R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{14}}$	$(1,0,0,0,1,2,2,2,2,2,2,1)$	2
$(7,4)$	$D3_5$	$\frac{R_3R_4R_5R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{14}}$	$(1,0,0,1,1,1,2,2,2,2,2,1)$	0
$(7,4)$	$D5_4$	$\frac{R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{14}}$	$(1,0,1,1,1,1,1,2,2,2,2,1)$	-2
$(7,5)$	$D7_3$	$\frac{R_1R_2R_3R_4R_5R_6R_7R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{14}}$	$(1,1,1,1,1,1,1,1,2,2,3,0)$	0
$(7,5)$	$D5_4$	$\frac{R_2R_3R_4R_5R_6R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{14}}$	$(1,0,1,1,1,1,1,2,2,2,3,0)$	2
$(8,2)$	$D5_5$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^2l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,0,4)$	2
$(8,3)$	$D5_5$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,1,3)$	-4

Table 24 — Continued on next page

Table 24 — Continued from previous page

Level (m_9, m_{10})	Dp_i	Mass	E_{12} Root (e_i basis)	Root length squared
$(8,3)$	$D1_7$	$\frac{R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{16}}$	$(1,0,0,1,2,2,2,2,2,2,1,3)$	0
$(8,3)$	$D3_6$	$\frac{R_2R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{16}}$	$(1,0,1,1,1,2,2,2,2,2,1,3)$	-2
$(8,4)$	$D5_5$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,2,2)$	-6
$(8,4)$	$D1_7$	$\frac{R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{16}}$	$(1,0,0,1,2,2,2,2,2,2,2,2)$	-2
$(8,4)$	$D3_6$	$\frac{R_2R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{16}}$	$(1,0,1,1,1,2,2,2,2,2,2,2)$	-4
$(8,5)$	$D5_5$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,3,1)$	-4
$(8,5)$	$D1_7$	$\frac{R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{16}}$	$(1,0,0,1,2,2,2,2,2,2,3,1)$	0
$(8,5)$	$D3_6$	$\frac{R_2R_3R_4R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{16}}$	$(1,0,1,1,1,2,2,2,2,2,3,1)$	-2
$(8,6)$	$D5_5$	$\frac{R_1R_2R_3R_4R_5R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^6l_s^{16}}$	$(1,1,1,1,1,1,2,2,2,2,4,0)$	2
$(9,3)$	$D3_7$	$\frac{R_1R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{18}}$	$(1,1,1,1,2,2,2,2,2,2,1,4)$	-4
$(9,3)$	$D1_8$	$\frac{R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{18}}$	$(1,0,1,2,2,2,2,2,2,2,1,4)$	-2
$(9,4)$	$D3_7$	$\frac{R_1R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{18}}$	$(1,1,1,1,2,2,2,2,2,2,2,3)$	-8
$(9,4)$	$D1_8$	$\frac{R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{18}}$	$(1,0,1,2,2,2,2,2,2,2,2,3)$	-6
$(9,5)$	$D3_7$	$\frac{R_1R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{18}}$	$(1,1,1,1,2,2,2,2,2,2,3,2)$	-8
$(9,5)$	$D1_8$	$\frac{R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{18}}$	$(1,0,1,2,2,2,2,2,2,2,3,2)$	-6
$(9,6)$	$D3_7$	$\frac{R_1R_2R_3R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^6l_s^{18}}$	$(1,1,1,1,2,2,2,2,2,2,4,1)$	-4
$(9,6)$	$D1_8$	$\frac{R_2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^6l_s^{18}}$	$(1,0,1,2,2,2,2,2,2,2,4,1)$	-2
$(10,3)$	$D1_9$	$\frac{R_1R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^3l_s^{20}}$	$(1,1,2,2,2,2,2,2,2,2,1,5)$	-4
$(10,4)$	$D1_9$	$\frac{R_1R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^4l_s^{20}}$	$(1,1,2,2,2,2,2,2,2,2,2,4)$	-10
$(10,5)$	$D1_9$	$\frac{R_1R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^5l_s^{20}}$	$(1,1,2,2,2,2,2,2,2,2,3,3)$	-12
$(10,6)$	$D1_9$	$\frac{R_1R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^6l_s^{20}}$	$(1,1,2,2,2,2,2,2,2,2,4,2)$	-10
$(10,7)$	$D1_9$	$\frac{R_1R_2^2R_3^2R_4^2R_5^2R_6^2R_7^2R_8^2R_9^2\hat{R}_{10}^2}{g_s^7l_s^{20}}$	$(1,1,2,2,2,2,2,2,2,2,5,1)$	-4

Table 24: The full set of IIB Dp_i branes from the l_1 representation of E_{11}.

4. Conclusion

In this paper we have identified the U-duality brane charge multiplets within the l_1 representation and given explicitly the weights associated to the particle, string and membrane multiplets when D=3,4,5,6,7,8 - this is a generalisation of the results of [11]. We have also introduced a tension formula that associates a tension to each root in the E_{12} root lattice, the lattice natural to the l_1 representation of E_{11}. The tension formula can be readily extended to other infinite dimensional algebras. The formula was constructed by introducing a dimensionful parameter into the l_1 representation of E_{11}, associated to the extra node (denoted in this paper with a "*") of the E_{12} Dynkin diagram that differentiates it from the E_{11} diagram. The tension formula reproduced the tensions of the pp-wave, the M2-brane, the M5-brane and the KK6 monopole of M-theory from the associated weights in the l_1 algebra. Furthermore all the tensions of the Dp-branes of IIA and IIB superstring theories were also found, together with the correct powers of the string coupling constant, g_s, and the string length, l_s. The tension formula was also applied to all the states in the particle and string charge multiplets for comparison with previously known results. The dimensionful parameter introduced was found to correctly reproduce all the known masses of the U-duality brane charge multiplets presented in [6−10]. The formula was then applied to the charge multiplet of the membrane and the corresponding masses were given. It would be illuminating to analyse the content of other G^{+++} algebras using the same construction. This would be especially interesting for the pure gravitational Kac-Moody algebra A^{+++}_{D-3} and the algebra associated to the bosonic string, $K_{27} \equiv D^{+++}_{24}$.

One consequence of the tension formula is the observation that almost all the content of E_{11} is associated to KK-branes (or monopoles). This interpretation is based on the observation that the tension of KK-branes is divergent when the spacetime is decompactified. Given the l_1 representation of E_{11} we can calculate for any of the charges an associated mass and tension. The tension is found by dividing the mass by a volume, $V_p \sim (2\pi R)^p$. For branes the tension is independent of a radius of compactification, in the cases where the tension remains dependent on the compact radii the associated solution is a KK-brane. A familiar example is generalisation of the Taub-NUT solution to eleven dimensions, which is also called the KK6-brane and is associated to the dual gravity field. The charge conserved by the dual gravity field appears in the l_1 representation at level three in the algebraic decomposition. The root in the E_{12} lattice is:

$$\beta_{KK6} \equiv \alpha_* + \alpha_1 + \cdots \alpha_4 + 2\alpha_5 + 3\alpha_6 + 4\alpha_7 + 5\alpha_8 + 3\alpha_9 + \alpha_{10} + 3\alpha_{11} \tag{4.1}$$

$$= (e_5 + \cdots e_{10} + 2e_{11}) \tag{4.2}$$

Using the formula of equation (3.10) the mass of this root is found.

$$Z_{\beta_{KK6}} = \frac{R_5 \ldots R_{10} R_{11}^2}{l_p^9} \tag{4.3}$$

For convenience we analyse the solution when it is compactified on a seven-torus and the tension of the solution may be found by dividing the mass by $(2\pi)^7 R_5 \ldots R_{11}$:

$$T_{\beta_{KK6}} = \frac{R_{11}}{(2\pi)^7 l_p^9} \tag{4.4}$$

Evidently this result does not carry naturally back into the uncompactified eleven dimensional spacetime and diverges when R_{11} is decompactified. The diverging tension in uncompactified spacetime is the signature of a KK-brane. For most of the weights of E_{11} appearing in the l_1 representation, in fact all those associated to a mixed symmetry field, the tension found by using equation (3.10) is divergent in the non-compact spacetime. In this sense much of the l_1 representation, being composed mostly of mixed symmetry fields, is associated to KK-brane charges. One may say that KK-brane charges are the rule and their vanishing in the case of the $M2$, $M5$ and pp-wave charges are the exceptions.

We have been able to classify the KK-brane charges that appear in the l_1 algebra since they are directly related to the parameter, k, appearing in the algebraic decomposition of section 1 of this paper. We recall that this parameter played a twofold role of counting the number of blocks of antisymmetric indices as well as controlling the blocks of eleven antisymmetric indices, or volume forms ϵ, appearing in the generators of the l_1 representation of E_{11}. The KK-branes may be labelled by the powers of the radii appearing in the mass formula. The most studied class of KK-branes, labelled Dp_i in the literature, have a mass which is quadratic in the spatial radii and may be labelled by two integers corresponding to the number of linear, p, and squared radii, i, respectively. In tables 22, 23 and 24 we list the full set of such Dp_i brane charges in the l_1 representation of E_{11} relevant to M-theory, the IIA theory and the IIB theory including the charges of KK-branes previously found by U-duality transformations of the D7 brane charge in [21]. The role played by KK-branes is unclear. A simple interpretation of the KK-brane charges of M-theory is that they give an eleven dimensional origin to the Dp-brane charges ($p > 5$) of IIA and IIB string theory upon dimensional reduction as well as other KK-brane charges. However, whether they play a more important part than simply book-keeping the branes, and also winding and KK-modes that appear from duality arguments in lower dimensional theories remains to be seen. One may hope that the extra KK-brane tower of states, being non-perturbative, may reveal significant details about the kinematics and dynamics of the E_{11} fields. More simply, since the prototype KK-brane is the dual graviton their further investigation may shed more light on the dual gravity theories.

An interesting infinite class of roots in the adjoint of E_{11} which corresponds to generators with no blocks of ten or eleven antisymmetrised indices has recently been completely found [20] and it was highlighted that E_{11} contained all the dualised versions of the tensors of massless dualised supergravity. The fields in this class of roots have associated generators taking the form,

$$K^{a_1^1\ldots a_9^1,\ldots a_1^n\ldots a_9^n,b}{}_c,\, R^{a_1^1\ldots a_9^1,\ldots a_1^n\ldots a_9^n,j_1j_2j_3},\, R^{a_1^1\ldots a_9^1,\ldots a_1^n\ldots a_9^n,j_1\ldots j_6},\, R^{a_1^1\ldots a_9^1,\ldots a_1^n\ldots a_9^n,j_1\ldots j_8,k} \tag{4.5}$$

Where $n \geq 0$. At first sight these generators, being all representations of the little group in eleven dimensions, SO(9), appear to indicate all possible massless dual solutions in the algebra, but if we consider the charges associated to these roots in the l_1 representation we find that they have different "masses", namely,

$$\frac{V^n}{l_p^{9n}R_{11}},\, \frac{V^nR_{10}R_{11}}{l_p^{9n+3}},\, \frac{V^nR_7R_8R_9R_{10}R_{11}}{l_p^{9n+6}},\, \frac{V^nR_5\ldots R_{10}R_{11}^2}{l_p^{9n+9}} \tag{4.6}$$

Where $V = R_3 \ldots R_{11}$. For the case where $n = 0$ three of the tensions, given by division by $(2\pi)^9V$, are well-defined and the fourth, which is the $KK6$ brane tension diverges. However for the other roots in this class ($n > 0$) the tensions all diverge in the non-compact setting which may indicate some difference in their nature. Using the techniques of [4, 13] one can write down a line element associated to a half-BPS brane solution for each of the dual roots in the adjoint of E_{11}. The set of roots are:

$$\beta_{pp*} = \alpha_{10} + n\beta_0 \tag{4.7}$$

$$\beta_{M2*} = \alpha_{11} + n\beta_0 \tag{4.8}$$

$$\beta_{M5*} = \alpha_6 + 2\alpha_7 + 3\alpha_8 + 2\alpha_9 + \alpha_{10} + 2\alpha_{11} + n\beta_0 \tag{4.9}$$

$$\beta_{KK6*} = \alpha_4 + 2\alpha_5 + 3\alpha_6 + 4\alpha_7 + 5\alpha_8 + 3\alpha_9 + \alpha_{10} + 3\alpha_{11} + n\beta_0 \tag{4.10}$$

Where,

$$\beta_0 \equiv \alpha_3 + 2\alpha_4 + 3\alpha_5 + 4\alpha_6 + 5\alpha_7 + 6\alpha_8 + 4\alpha_9 + 2\alpha_{10} + 3\alpha_{11} \tag{4.11}$$

is the root controlling the blocks of nine antisymmetric indices in the dual generators. For reference,

$$Z_{\beta_0} = \frac{R_3 \dots R_{11}}{l_p^9} = \frac{R_1 R_2 R_3}{G_{11}} \tag{4.12}$$

The line elements corresponding to the set of dual roots are,

$$ds^2_{pp*} = (1+K)^n d\bar{x}_2^2 - (1-K)(dt^2) + (1+K)dy^2 - 2K dt dy + d\Sigma_7^2 \tag{4.13}$$

$$ds^2_{M2*} = N^{(n+\frac{1}{3})}(d\bar{x}_2^2) + N^{\frac{1}{3}}(d\bar{x}_6^2) + N^{-\frac{2}{3}}(d\bar{y}_2^2 - dt^2) \tag{4.14}$$

$$ds^2_{M5*} = N^{(n+\frac{2}{3})}(d\bar{x}_2^2) + N^{\frac{2}{3}}(d\bar{x}_3^2) + N^{-\frac{1}{3}}(d\bar{y}_5^2 - dt^2) \tag{4.15}$$

$$ds^2_{KK6*} = N^{(n+1)}(d\bar{x}_2^2) + N(dy^2) + N^{-1}(-dt^2) + d\Sigma_7^2 \tag{4.16}$$

Where $d\Sigma_7^2$ denotes a seven dimensional Euclidean line element. These volume elements are dependent on n, the number of blocks of nine antisymmetric indices appearing in the generator of the associated dual roots. It would be interesting to study this class of roots further.

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A. Low level weights in the l_1 representation of E_{11} relevant to 11D, IIA and IIB SuGra

Level (m_{11})	A_{10} weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
0	[1, 0, 0, 0, 0, 0, 0, 0, 0, 0]	(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(1, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	2
1	[0, 0, 0, 0, 0, 0, 0, 0, 1, 0]	(1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 1)	(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1)	2
2	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0]	(1, 1, 1, 1, 1, 1, 1, 2, 3, 2, 1, 2)	(1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1)	2
3	[0, 0, 1, 0, 0, 0, 0, 0, 0, 0]	(1, 1, 1, 1, 2, 3, 4, 5, 6, 4, 2, 3)	(1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1)	0
3	[0, 0, 0, 1, 0, 0, 0, 0, 0, 1]	(1, 1, 1, 1, 1, 2, 3, 4, 5, 3, 1, 3)	(1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2)	2
4	[0, 0, 1, 0, 0, 0, 0, 1, 0, 0]	(1, 1, 1, 1, 2, 3, 4, 5, 6, 4, 2, 4)	(1, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2)	2
4	[0, 1, 0, 0, 0, 0, 0, 0, 1, 0]	(1, 1, 1, 2, 3, 4, 5, 6, 7, 4, 2, 4)	(1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 2, 2)	0
4	[1, 0, 0, 0, 0, 0, 0, 0, 0, 1]	(1, 1, 2, 3, 4, 5, 6, 7, 8, 5, 2, 4)	(1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2)	−2
4	[0, 1, 0, 0, 0, 0, 0, 0, 0, 2]	(1, 1, 1, 2, 3, 4, 5, 6, 7, 4, 1, 4)	(1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 3)	2
5	[0, 0, 0, 0, 0, 0, 0, 1, 0, 0]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 6, 3, 5)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2)	−4
5	[0, 0, 1, 0, 1, 0, 0, 0, 0, 0]	(1, 1, 1, 1, 2, 3, 5, 7, 9, 6, 3, 5)	(1, 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 2)	2
5	[0, 1, 0, 0, 0, 1, 0, 0, 0, 0]	(1, 1, 1, 2, 3, 4, 5, 7, 9, 6, 3, 5)	(1, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2)	0
5	[1, 0, 0, 0, 0, 0, 1, 0, 0, 0]	(1, 1, 2, 3, 4, 5, 6, 7, 9, 6, 3, 5)	(1, 0, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2)	−2
5	[0, 0, 0, 0, 0, 0, 0, 0, 1, 1]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 5, 2, 5)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 3)	−2
5	[0, 1, 0, 0, 0, 0, 1, 0, 0, 1]	(1, 1, 1, 2, 3, 4, 5, 6, 8, 5, 2, 5)	(1, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 3)	2
5	[1, 0, 0, 0, 0, 0, 0, 0, 2, 0]	(1, 1, 2, 3, 4, 5, 6, 7, 8, 4, 2, 5)	(1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3)	2
5	[1, 0, 0, 0, 0, 0, 0, 1, 0, 1]	(1, 1, 2, 3, 4, 5, 6, 7, 8, 5, 2, 5)	(1, 0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 3)	0
5	[0, 0, 0, 0, 0, 0, 0, 0, 0, 3]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 5, 1, 5)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 4)	2
6	[0, 0, 0, 0, 1, 0, 0, 0, 0, 0]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 8, 4, 6)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2)	−6
6	[0, 1, 1, 0, 0, 0, 0, 0, 0, 0]	(1, 1, 1, 2, 4, 6, 8, 10, 12, 8, 4, 6)	(1, 0, 0, 1, 2, 2, 2, 2, 2, 2, 2, 2)	−2
6	[1, 0, 0, 1, 0, 0, 0, 0, 0, 0]	(1, 1, 2, 3, 4, 6, 8, 10, 12, 8, 4, 6)	(1, 0, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2)	−4
6	[0, 0, 0, 0, 0, 0, 0, 2, 0, 0]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 6, 3, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3)	0
6	[0, 0, 0, 0, 0, 0, 1, 0, 1, 0]	(1, 2, 3, 4, 5, 6, 7, 8, 10, 6, 3, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3)	−2
6	[0, 0, 0, 0, 0, 1, 0, 0, 0, 1]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 7, 3, 6)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3)	−4
6	[0, 0, 2, 0, 0, 0, 0, 0, 0, 1]	(1, 1, 1, 1, 3, 5, 7, 9, 11, 7, 3, 6)	(1, 0, 0, 0, 2, 2, 2, 2, 2, 2, 2, 3)	2
6	[0, 1, 0, 0, 1, 0, 0, 0, 1, 0]	(1, 1, 1, 2, 3, 4, 6, 8, 10, 6, 3, 6)	(1, 0, 0, 1, 1, 1, 2, 2, 2, 2, 3, 3)	2
6	[0, 1, 0, 1, 0, 0, 0, 0, 0, 1]	(1, 1, 1, 2, 3, 5, 7, 9, 11, 7, 3, 6)	(1, 0, 0, 1, 1, 2, 2, 2, 2, 2, 2, 3)	0
6	[1, 0, 0, 0, 0, 0, 1, 1, 0, 0]	(1, 1, 2, 3, 4, 5, 6, 7, 9, 6, 3, 6)	(1, 0, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3)	2
6	[1, 0, 0, 0, 0, 1, 0, 0, 1, 0]	(1, 1, 2, 3, 4, 5, 6, 8, 10, 6, 3, 6)	(1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3)	0
6	[1, 0, 0, 0, 1, 0, 0, 0, 0, 1]	(1, 1, 2, 3, 4, 5, 7, 9, 11, 7, 3, 6)	(1, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3)	−2
6	[0, 0, 0, 0, 0, 0, 0, 1, 1, 1]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 5, 2, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 3, 4)	2
6	[0, 0, 0, 0, 0, 0, 1, 0, 0, 2]	(1, 2, 3, 4, 5, 6, 7, 8, 10, 6, 2, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 4)	0
6	[1, 0, 0, 0, 0, 1, 0, 0, 0, 2]	(1, 1, 2, 3, 4, 5, 6, 8, 10, 6, 2, 6)	(1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 4)	2
7	[0, 1, 0, 0, 0, 0, 0, 0, 0, 0]	(1, 2, 3, 5, 7, 9, 11, 13, 15, 10, 5, 7)	(1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2)	−10
7	[2, 0, 0, 0, 0, 0, 0, 0, 0, 0]	(1, 1, 3, 5, 7, 9, 11, 13, 15, 10, 5, 7)	(1, 0, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)	−8
7	[0, 0, 0, 0, 0, 1, 1, 0, 0, 0]	(1, 2, 3, 4, 5, 6, 7, 9, 12, 8, 4, 7)	(1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3, 3)	−2
7	[0, 0, 0, 0, 1, 0, 0, 1, 0, 0]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 8, 4, 7)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3)	−4
7	[0, 0, 0, 1, 0, 0, 0, 0, 1, 0]	(1, 2, 3, 4, 5, 7, 9, 11, 13, 8, 4, 7)	(1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3)	−6
7	[0, 0, 1, 0, 0, 0, 0, 0, 0, 1]	(1, 2, 3, 4, 6, 8, 10, 12, 14, 9, 4, 7)	(1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3)	−8
7	[0, 1, 0, 1, 0, 0, 1, 0, 0, 0]	(1, 1, 1, 2, 3, 5, 7, 9, 12, 8, 4, 7)	(1, 0, 0, 1, 1, 2, 2, 2, 3, 3, 3, 3)	2
7	[0, 1, 1, 0, 0, 0, 0, 1, 0, 0]	(1, 1, 1, 2, 4, 6, 8, 10, 12, 8, 4, 7)	(1, 0, 0, 1, 2, 2, 2, 2, 2, 3, 3, 3)	0
7	[0, 2, 0, 0, 0, 0, 0, 0, 1, 0]	(1, 1, 1, 3, 5, 7, 9, 11, 13, 8, 4, 7)	(1, 0, 0, 2, 2, 2, 2, 2, 2, 2, 3, 3)	−2
7	[1, 0, 0, 0, 0, 2, 0, 0, 0, 0]	(1, 1, 2, 3, 4, 5, 6, 9, 12, 8, 4, 7)	(1, 0, 1, 1, 1, 1, 1, 3, 3, 3, 3, 3)	2
7	[1, 0, 0, 0, 1, 0, 1, 0, 0, 0]	(1, 1, 2, 3, 4, 5, 7, 9, 12, 8, 4, 7)	(1, 0, 1, 1, 1, 1, 2, 2, 3, 3, 3, 3)	0
7	[1, 0, 0, 1, 0, 0, 0, 1, 0, 0]	(1, 1, 2, 3, 4, 6, 8, 10, 12, 8, 4, 7)	(1, 0, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3)	−2
7	[1, 0, 1, 0, 0, 0, 0, 0, 1, 0]	(1, 1, 2, 3, 5, 7, 9, 11, 13, 8, 4, 7)	(1, 0, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3)	−4
7	[1, 1, 0, 0, 0, 0, 0, 0, 0, 1]	(1, 1, 2, 4, 6, 8, 10, 12, 14, 9, 4, 7)	(1, 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 3)	−6
7	[0, 0, 0, 0, 0, 0, 2, 0, 0, 1]	(1, 2, 3, 4, 5, 6, 7, 8, 11, 7, 3, 7)	(1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 4)	2
7	[0, 0, 0, 0, 0, 1, 0, 0, 2, 0]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 6, 3, 7)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 4, 4)	2
7	[0, 0, 0, 0, 0, 1, 0, 1, 0, 1]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 7, 3, 7)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, 4)	0
7	[0, 0, 0, 0, 1, 0, 0, 0, 1, 1]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 7, 3, 7)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 4)	−2
7	[0, 0, 0, 1, 0, 0, 0, 0, 0, 2]	(1, 2, 3, 4, 5, 7, 9, 11, 13, 8, 3, 7)	(1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 4)	−4
7	[0, 1, 1, 0, 0, 0, 0, 0, 1, 1]	(1, 1, 1, 2, 4, 6, 8, 10, 12, 7, 3, 7)	(1, 0, 0, 1, 2, 2, 2, 2, 2, 2, 3, 4)	2

Table 25 — Continued on next page

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Table 25 — Continued from previous page

Level (m_{11})	A_{10} weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
7	[0,2,0,0,0,0,0,0,0,2]	(1,1,1,3,5,7,9,11,13,8,3,7)	(1,0,0,2,2,2,2,2,2,2,2,4)	0
7	[1,0,0,0,1,0,0,1,0,1]	(1,1,2,3,4,5,7,9,11,7,3,7)	(1,0,1,1,1,1,2,2,2,3,3,4)	2
7	[1,0,0,1,0,0,0,0,1,1]	(1,1,2,3,4,6,8,10,12,7,3,7)	(1,0,1,1,1,2,2,2,2,2,3,4)	0
7	[1,0,1,0,0,0,0,0,0,2]	(1,1,2,3,5,7,9,11,13,8,3,7)	(1,0,1,1,2,2,2,2,2,2,2,4)	−2
7	[0,0,0,0,1,0,0,0,0,3]	(1,2,3,4,5,6,8,10,12,7,2,7)	(1,1,1,1,1,1,2,2,2,2,2,5)	2
8	[0,0,0,0,0,0,0,0,0,1]	(1,3,5,7,9,11,13,15,17,11,5,8)	(1,2,2,2,2,2,2,2,2,2,2,3)	−14
8	[0,0,0,0,2,0,0,0,0,0]	(1,2,3,4,5,6,9,12,15,10,5,8)	(1,1,1,1,1,1,3,3,3,3,3,3)	−4
8	[0,0,0,1,0,1,0,0,0,0]	(1,2,3,4,5,7,9,12,15,10,5,8)	(1,1,1,1,1,2,2,3,3,3,3,3)	−6
8	[0,0,1,0,0,0,1,0,0,0]	(1,2,3,4,6,8,10,12,15,10,5,8)	(1,1,1,1,2,2,2,2,3,3,3,3)	−8
8	[0,1,0,0,0,0,0,1,0,0]	(1,2,3,5,7,9,11,13,15,10,5,8)	(1,1,1,2,2,2,2,2,2,3,3,3)	−10
8	[0,1,0,2,0,0,0,0,0,0]	(1,1,1,2,3,6,9,12,15,10,5,8)	(1,0,0,1,1,3,3,3,3,3,3,3)	2
8	[0,1,1,0,1,0,0,0,0,0]	(1,1,1,2,4,6,9,12,15,10,5,8)	(1,0,0,1,2,2,3,3,3,3,3,3)	0
8	[0,2,0,0,0,1,0,0,0,0]	(1,1,1,3,5,7,9,12,15,10,5,8)	(1,0,0,2,2,2,2,3,3,3,3,3)	−2
8	[1,0,0,0,0,0,0,0,1,0]	(1,2,4,6,8,10,12,14,16,10,5,8)	(1,1,2,2,2,2,2,2,2,2,3,3)	−12
8	[1,0,0,1,1,0,0,0,0,0]	(1,1,2,3,4,6,9,12,15,10,5,8)	(1,0,1,1,1,2,3,3,3,3,3,3)	−2
8	[1,0,1,0,0,1,0,0,0,0]	(1,1,2,3,5,7,9,12,15,10,5,8)	(1,0,1,1,2,2,2,3,3,3,3,3)	−4
8	[1,1,0,0,0,0,1,0,0,0]	(1,1,2,4,6,8,10,12,15,10,5,8)	(1,0,1,2,2,2,2,2,3,3,3,3)	−6
8	[2,0,0,0,0,0,0,1,0,0]	(1,1,3,5,7,9,11,13,15,10,5,8)	(1,0,2,2,2,2,2,2,2,3,3,3)	−8
8	[0,0,0,0,0,2,0,0,1,0]	(1,2,3,4,5,6,7,10,13,8,4,8)	(1,1,1,1,1,1,1,3,3,3,4,4)	2
8	[0,0,0,0,1,0,0,2,0,0]	(1,2,3,4,5,6,8,10,12,8,4,8)	(1,1,1,1,1,1,2,2,2,4,4,4)	2
8	[0,0,0,0,1,0,1,0,1,0]	(1,2,3,4,5,6,8,10,13,8,4,8)	(1,1,1,1,1,1,2,2,3,3,4,4)	0
8	[0,0,0,0,1,1,0,0,0,1]	(1,2,3,4,5,6,8,11,14,9,4,8)	(1,1,1,1,1,1,2,3,3,3,3,4)	−2
8	[0,0,0,1,0,0,0,1,1,0]	(1,2,3,4,5,7,9,11,13,8,4,8)	(1,1,1,1,1,2,2,2,2,3,4,4)	−2
8	[0,0,0,1,0,0,1,0,0,1]	(1,2,3,4,5,7,9,11,14,9,4,8)	(1,1,1,1,1,2,2,2,3,3,3,4)	−4
8	[0,0,1,0,0,0,0,0,2,0]	(1,2,3,4,6,8,10,12,14,8,4,8)	(1,1,1,1,2,2,2,2,2,2,4,4)	−4
8	[0,0,1,0,0,0,0,1,0,1]	(1,2,3,4,6,8,10,12,14,9,4,8)	(1,1,1,1,2,2,2,2,2,3,3,4)	−6
8	[0,1,0,0,0,0,0,0,1,1]	(1,2,3,5,7,9,11,13,15,9,4,8)	(1,1,1,2,2,2,2,2,2,2,3,4)	−8
8	[0,1,1,0,0,1,0,0,0,1]	(1,1,1,2,4,6,8,11,14,9,4,8)	(1,0,0,1,2,2,2,3,3,3,3,4)	2
8	[0,2,0,0,0,0,0,1,1,0]	(1,1,1,3,5,7,9,11,13,8,4,8)	(1,0,0,2,2,2,2,2,2,3,4,4)	2
8	[0,2,0,0,0,0,1,0,0,1]	(1,1,1,3,5,7,9,11,14,9,4,8)	(1,0,0,2,2,2,2,2,3,3,3,4)	0
8	[1,0,0,0,0,0,0,0,0,2]	(1,2,4,6,8,10,12,14,16,10,4,8)	(1,1,2,2,2,2,2,2,2,2,2,4)	−10
8	[1,0,0,0,2,0,0,0,0,1]	(1,1,2,3,4,5,8,11,14,9,4,8)	(1,0,1,1,1,1,3,3,3,3,3,4)	2
8	[1,0,0,1,0,0,1,0,1,0]	(1,1,2,3,4,6,8,10,13,8,4,8)	(1,0,1,1,1,2,2,2,3,3,4,4)	2
8	[1,0,0,1,0,1,0,0,0,1]	(1,1,2,3,4,6,8,11,14,9,4,8)	(1,0,1,1,1,2,2,3,3,3,3,4)	0
8	[1,0,1,0,0,0,0,1,1,0]	(1,1,2,3,5,7,9,11,13,8,4,8)	(1,0,1,1,2,2,2,2,2,3,4,4)	0
8	[1,0,1,0,0,0,1,0,0,1]	(1,1,2,3,5,7,9,11,14,9,4,8)	(1,0,1,1,2,2,2,2,3,3,3,4)	−2
8	[1,1,0,0,0,0,0,0,2,0]	(1,1,2,4,6,8,10,12,14,8,4,8)	(1,0,1,2,2,2,2,2,2,2,4,4)	−2
8	[1,1,0,0,0,0,0,1,0,1]	(1,1,2,4,6,8,10,12,14,9,4,8)	(1,0,1,2,2,2,2,2,2,3,3,4)	−4
8	[2,0,0,0,0,0,0,0,1,1]	(1,1,3,5,7,9,11,13,15,9,4,8)	(1,0,2,2,2,2,2,2,2,2,3,4)	−6
8	[0,0,0,0,1,0,1,0,0,2]	(1,2,3,4,5,6,8,10,13,8,3,8)	(1,1,1,1,1,1,2,2,3,3,3,5)	2
8	[0,0,0,1,0,0,0,0,2,1]	(1,2,3,4,5,7,9,11,13,7,3,8)	(1,1,1,1,1,2,2,2,2,2,4,5)	2
8	[0,0,0,1,0,0,0,1,0,2]	(1,2,3,4,5,7,9,11,13,8,3,8)	(1,1,1,1,1,2,2,2,2,3,3,5)	0
8	[0,0,1,0,0,0,0,0,1,2]	(1,2,3,4,6,8,10,12,14,8,3,8)	(1,1,1,1,2,2,2,2,2,2,3,5)	−2

Table 25: Low level weights in the *11D supergravity* decomposition of the l_1 representation of E_{11}

Level (m_{10}, m_{11})	A_9 weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
(0,0)	[1,0,0,0,0,0,0,0,0]	(1,0,0,0,0,0,0,0,0,0,0,0)	(1,−1,0,0,0,0,0,0,0,0,0,0)	2
(1,0)	[0,0,0,0,0,0,0,0,0]	(1,1,1,1,1,1,1,1,1,1,1,0)	(1,0,0,0,0,0,0,0,0,0,0,−1)	2
(0,1)	[0,0,0,0,0,0,0,0,1]	(1,1,1,1,1,1,1,1,1,0,0,1)	(1,0,0,0,0,0,0,0,0,0,1,1)	2
(1,1)	[0,0,0,0,0,0,0,1,0]	(1,1,1,1,1,1,1,1,1,1,1,1)	(1,0,0,0,0,0,0,0,0,1,1,0)	2
(1,2)	[0,0,0,0,0,1,0,0,0]	(1,1,1,1,1,1,1,2,3,2,1,2)	(1,0,0,0,0,0,0,1,1,1,1,1)	2

Table 26 — Continued on next page

Table 26 — Continued from previous page

Level (m_{10}, m_{11})	A_9 weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
$(2,2)$	$[0,0,0,0,1,0,0,0,0]$	$(1,1,1,1,1,1,2,3,4,3,2,2)$	$(1,0,0,0,0,0,1,1,1,1,1,0)$	2
$(1,3)$	$[0,0,0,1,0,0,0,0,0]$	$(1,1,1,1,1,2,3,4,5,3,1,3)$	$(1,0,0,0,0,1,1,1,1,1,1,2)$	2
$(2,3)$	$[0,0,1,0,0,0,0,0,0]$	$(1,1,1,1,2,3,4,5,6,4,2,3)$	$(1,0,0,0,1,1,1,1,1,1,1,1)$	0
$(2,3)$	$[0,0,0,1,0,0,0,0,1]$	$(1,1,1,1,1,2,3,4,5,3,2,3)$	$(1,0,0,0,0,1,1,1,1,1,2,1)$	2
$(3,3)$	$[0,1,0,0,0,0,0,0,0]$	$(1,1,1,2,3,4,5,6,7,5,3,3)$	$(1,0,0,1,1,1,1,1,1,1,1,0)$	0
$(3,3)$	$[0,0,1,0,0,0,0,0,1]$	$(1,1,1,1,2,3,4,5,6,4,3,3)$	$(1,0,0,0,1,1,1,1,1,1,2,0)$	2
$(4,3)$	$[1,0,0,0,0,0,0,0,0]$	$(1,1,2,3,4,5,6,7,8,6,4,3)$	$(1,0,1,1,1,1,1,1,1,1,1,-1)$	2
$(1,4)$	$[0,1,0,0,0,0,0,0,0]$	$(1,1,1,2,3,4,5,6,7,4,1,4)$	$(1,0,0,1,1,1,1,1,1,1,1,3)$	2
$(2,4)$	$[1,0,0,0,0,0,0,0,0]$	$(1,1,2,3,4,5,6,7,8,5,2,4)$	$(1,0,1,1,1,1,1,1,1,1,1,2)$	-2
$(2,4)$	$[0,0,1,0,0,0,0,1,0]$	$(1,1,1,1,2,3,4,5,6,4,2,4)$	$(1,0,0,0,1,1,1,1,1,2,2,2)$	2
$(2,4)$	$[0,1,0,0,0,0,0,0,1]$	$(1,1,1,2,3,4,5,6,7,4,2,4)$	$(1,0,0,1,1,1,1,1,1,1,2,2)$	0
$(3,4)$	$[0,0,0,0,0,0,0,0,0]$	$(1,2,3,4,5,6,7,8,9,6,3,4)$	$(1,1,1,1,1,1,1,1,1,1,1,1)$	-4
$(3,4)$	$[0,0,1,0,0,0,1,0,0]$	$(1,1,1,1,2,3,4,5,7,5,3,4)$	$(1,0,0,0,1,1,1,1,2,2,2,1)$	2
$(3,4)$	$[0,1,0,0,0,0,0,1,0]$	$(1,1,1,2,3,4,5,6,7,5,3,4)$	$(1,0,0,1,1,1,1,1,1,2,2,1)$	0
$(3,4)$	$[1,0,0,0,0,0,0,0,1]$	$(1,1,2,3,4,5,6,7,8,5,3,4)$	$(1,0,1,1,1,1,1,1,1,1,2,1)$	-2
$(3,4)$	$[0,1,0,0,0,0,0,0,2]$	$(1,1,1,2,3,4,5,6,7,4,3,4)$	$(1,0,0,1,1,1,1,1,1,1,3,1)$	2
$(4,4)$	$[0,0,0,0,0,0,0,0,1]$	$(1,2,3,4,5,6,7,8,9,6,4,4)$	$(1,1,1,1,1,1,1,1,1,1,2,0)$	-2
$(4,4)$	$[0,1,0,0,0,0,1,0,0]$	$(1,1,1,2,3,4,5,6,8,6,4,4)$	$(1,0,0,1,1,1,1,1,2,2,2,0)$	2
$(4,4)$	$[1,0,0,0,0,0,0,1,0]$	$(1,1,2,3,4,5,6,7,8,6,4,4)$	$(1,0,1,1,1,1,1,1,1,2,2,0)$	0
$(4,4)$	$[1,0,0,0,0,0,0,0,2]$	$(1,1,2,3,4,5,6,7,8,5,4,4)$	$(1,0,1,1,1,1,1,1,1,1,3,0)$	2
$(5,4)$	$[0,0,0,0,0,0,0,1,0]$	$(1,2,3,4,5,6,7,8,9,7,5,4)$	$(1,1,1,1,1,1,1,1,1,2,2,-1)$	2
$(1,5)$	$[0,0,0,0,0,0,0,0,0]$	$(1,2,3,4,5,6,7,8,9,5,1,5)$	$(1,1,1,1,1,1,1,1,1,1,1,4)$	2
$(2,5)$	$[0,0,0,0,0,0,0,0,1]$	$(1,2,3,4,5,6,7,8,9,5,2,5)$	$(1,1,1,1,1,1,1,1,1,1,2,3)$	-2
$(2,5)$	$[0,1,0,0,0,0,1,0,0]$	$(1,1,1,2,3,4,5,6,8,5,2,5)$	$(1,0,0,1,1,1,1,1,2,2,2,3)$	2
$(2,5)$	$[1,0,0,0,0,0,0,1,0]$	$(1,1,2,3,4,5,6,7,8,5,2,5)$	$(1,0,1,1,1,1,1,1,1,2,2,3)$	0
$(2,5)$	$[1,0,0,0,0,0,0,0,2]$	$(1,1,2,3,4,5,6,7,8,4,2,5)$	$(1,0,1,1,1,1,1,1,1,1,3,3)$	2
$(3,5)$	$[0,0,0,0,0,0,0,1,0]$	$(1,2,3,4,5,6,7,8,9,6,3,5)$	$(1,1,1,1,1,1,1,1,1,2,2,2)$	-4
$(3,5)$	$[0,0,1,0,1,0,0,0,0]$	$(1,1,1,1,2,3,5,7,9,6,3,5)$	$(1,0,0,0,1,1,2,2,2,2,2,2)$	2
$(3,5)$	$[0,1,0,0,0,1,0,0,0]$	$(1,1,1,2,3,4,5,7,9,6,3,5)$	$(1,0,0,1,1,1,1,2,2,2,2,2)$	0
$(3,5)$	$[1,0,0,0,0,0,1,0,0]$	$(1,1,2,3,4,5,6,7,9,6,3,5)$	$(1,0,1,1,1,1,1,1,2,2,2,2)$	-2
$(3,5)$	$[0,0,0,0,0,0,0,0,2]$	$(1,2,3,4,5,6,7,8,9,5,3,5)$	$(1,1,1,1,1,1,1,1,1,1,3,2)$	-2
$(3,5)$	$[0,1,0,0,0,0,1,0,1]$	$(1,1,1,2,3,4,5,6,8,5,3,5)$	$(1,0,0,1,1,1,1,1,2,2,3,2)$	2
$(3,5)$	$[1,0,0,0,0,0,0,1,1]$	$(1,1,2,3,4,5,6,7,8,5,3,5)$	$(1,0,1,1,1,1,1,1,1,2,3,2)$	0
$(4,5)$	$[0,0,0,0,0,0,1,0,0]$	$(1,2,3,4,5,6,7,8,10,7,4,5)$	$(1,1,1,1,1,1,1,1,2,2,2,1)$	-4
$(4,5)$	$[0,0,1,1,0,0,0,0,0]$	$(1,1,1,1,2,4,6,8,10,7,4,5)$	$(1,0,0,0,1,2,2,2,2,2,2,1)$	2
$(4,5)$	$[0,1,0,0,1,0,0,0,0]$	$(1,1,1,2,3,4,6,8,10,7,4,5)$	$(1,0,0,1,1,1,2,2,2,2,2,1)$	0
$(4,5)$	$[1,0,0,0,0,1,0,0,0]$	$(1,1,2,3,4,5,6,8,10,7,4,5)$	$(1,0,1,1,1,1,1,2,2,2,2,1)$	-2
$(4,5)$	$[0,0,0,0,0,0,0,1,1]$	$(1,2,3,4,5,6,7,8,9,6,4,5)$	$(1,1,1,1,1,1,1,1,1,2,3,1)$	-2
$(4,5)$	$[0,1,0,0,0,1,0,0,1]$	$(1,1,1,2,3,4,5,7,9,6,4,5)$	$(1,0,0,1,1,1,1,2,2,2,3,1)$	2
$(4,5)$	$[1,0,0,0,0,0,0,2,0]$	$(1,1,2,3,4,5,6,7,8,6,4,5)$	$(1,0,1,1,1,1,1,1,1,3,3,1)$	2
$(4,5)$	$[1,0,0,0,0,0,1,0,1]$	$(1,1,2,3,4,5,6,7,9,6,4,5)$	$(1,0,1,1,1,1,1,1,2,2,3,1)$	0
$(4,5)$	$[0,0,0,0,0,0,0,0,3]$	$(1,2,3,4,5,6,7,8,9,5,4,5)$	$(1,1,1,1,1,1,1,1,1,1,4,1)$	2
$(5,5)$	$[0,0,0,0,0,1,0,0,0]$	$(1,2,3,4,5,6,7,9,11,8,5,5)$	$(1,1,1,1,1,1,1,2,2,2,2,0)$	-2
$(5,5)$	$[0,1,0,1,0,0,0,0,0]$	$(1,1,1,2,3,5,7,9,11,8,5,5)$	$(1,0,0,1,1,2,2,2,2,2,2,0)$	2
$(5,5)$	$[1,0,0,0,1,0,0,0,0]$	$(1,1,2,3,4,5,7,9,11,8,5,5)$	$(1,0,1,1,1,1,2,2,2,2,2,0)$	0
$(5,5)$	$[0,0,0,0,0,0,0,2,0]$	$(1,2,3,4,5,6,7,8,9,7,5,5)$	$(1,1,1,1,1,1,1,1,1,3,3,0)$	2
$(5,5)$	$[0,0,0,0,0,0,1,0,1]$	$(1,2,3,4,5,6,7,8,10,7,5,5)$	$(1,1,1,1,1,1,1,1,2,2,3,0)$	0
$(5,5)$	$[1,0,0,0,0,1,0,0,1]$	$(1,1,2,3,4,5,6,8,10,7,5,5)$	$(1,0,1,1,1,1,1,2,2,2,3,0)$	2
$(6,5)$	$[0,0,0,0,1,0,0,0,0]$	$(1,2,3,4,5,6,8,10,12,9,6,5)$	$(1,1,1,1,1,1,2,2,2,2,2,-1)$	2
$(2,6)$	$[0,0,0,0,0,0,1,0,0]$	$(1,2,3,4,5,6,7,8,10,6,2,6)$	$(1,1,1,1,1,1,1,1,2,2,2,4)$	0
$(2,6)$	$[1,0,0,0,0,1,0,0,0]$	$(1,1,2,3,4,5,6,8,10,6,2,6)$	$(1,0,1,1,1,1,1,2,2,2,2,4)$	2
$(2,6)$	$[0,0,0,0,0,0,0,1,1]$	$(1,2,3,4,5,6,7,8,9,5,2,6)$	$(1,1,1,1,1,1,1,1,1,2,3,4)$	2
$(3,6)$	$[0,0,0,0,0,1,0,0,0]$	$(1,2,3,4,5,6,7,9,11,7,3,6)$	$(1,1,1,1,1,1,1,2,2,2,2,3)$	-4
$(3,6)$	$[0,0,2,0,0,0,0,0,0]$	$(1,1,1,1,3,5,7,9,11,7,3,6)$	$(1,0,0,0,2,2,2,2,2,2,2,3)$	2
$(3,6)$	$[0,1,0,1,0,0,0,0,0]$	$(1,1,1,2,3,5,7,9,11,7,3,6)$	$(1,0,0,1,1,2,2,2,2,2,2,3)$	0

Table 26 — Continued on next page

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Table 26 — Continued from previous page

Level (m_{10}, m_{11})	A_9 weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
(3, 6)	[1, 0, 0, 0, 1, 0, 0, 0, 0]	(1, 1, 2, 3, 4, 5, 7, 9, 11, 7, 3, 6)	(1, 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3)	−2
(3, 6)	[0, 0, 0, 0, 0, 0, 0, 2, 0]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 6, 3, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3)	0
(3, 6)	[0, 0, 0, 0, 0, 0, 1, 0, 1]	(1, 2, 3, 4, 5, 6, 7, 8, 10, 6, 3, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3)	−2
(3, 6)	[0, 1, 0, 0, 1, 0, 0, 0, 1]	(1, 1, 1, 2, 3, 4, 6, 8, 10, 6, 3, 6)	(1, 0, 0, 1, 1, 1, 2, 2, 2, 2, 3, 3)	2
(3, 6)	[1, 0, 0, 0, 0, 0, 1, 1, 0]	(1, 1, 2, 3, 4, 5, 6, 7, 9, 6, 3, 6)	(1, 0, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3)	2
(3, 6)	[1, 0, 0, 0, 0, 1, 0, 0, 1]	(1, 1, 2, 3, 4, 5, 6, 8, 10, 6, 3, 6)	(1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3)	0
(3, 6)	[0, 0, 0, 0, 0, 0, 0, 1, 2]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 5, 3, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 4, 3)	2
(4, 6)	[0, 0, 0, 0, 1, 0, 0, 0, 0]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 8, 4, 6)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2)	−6
(4, 6)	[0, 1, 1, 0, 0, 0, 0, 0, 0]	(1, 1, 1, 2, 4, 6, 8, 10, 12, 8, 4, 6)	(1, 0, 0, 1, 2, 2, 2, 2, 2, 2, 2, 2)	−2
(4, 6)	[1, 0, 0, 1, 0, 0, 0, 0, 0]	(1, 1, 2, 3, 4, 6, 8, 10, 12, 8, 4, 6)	(1, 0, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2)	−4
(4, 6)	[0, 0, 0, 0, 0, 0, 1, 1, 0]	(1, 2, 3, 4, 5, 6, 7, 8, 10, 7, 4, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 2)	−2
(4, 6)	[0, 0, 0, 0, 0, 1, 0, 0, 1]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 7, 4, 6)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 2)	−4
(4, 6)	[0, 0, 2, 0, 0, 0, 0, 0, 1]	(1, 1, 1, 1, 3, 5, 7, 9, 11, 7, 4, 6)	(1, 0, 0, 0, 2, 2, 2, 2, 2, 2, 3, 2)	2
(4, 6)	[0, 1, 0, 0, 1, 0, 0, 1, 0]	(1, 1, 1, 2, 3, 4, 6, 8, 10, 7, 4, 6)	(1, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 2)	2
(4, 6)	[0, 1, 0, 1, 0, 0, 0, 0, 1]	(1, 1, 1, 2, 3, 5, 7, 9, 11, 7, 4, 6)	(1, 0, 0, 1, 1, 2, 2, 2, 2, 2, 3, 2)	0
(4, 6)	[1, 0, 0, 0, 0, 0, 2, 0, 0]	(1, 1, 2, 3, 4, 5, 6, 7, 10, 7, 4, 6)	(1, 0, 1, 1, 1, 1, 1, 1, 3, 3, 3, 2)	2
(4, 6)	[1, 0, 0, 0, 0, 1, 0, 1, 0]	(1, 1, 2, 3, 4, 5, 6, 8, 10, 7, 4, 6)	(1, 0, 1, 1, 1, 1, 1, 2, 2, 3, 3, 2)	0
(4, 6)	[1, 0, 0, 0, 1, 0, 0, 0, 1]	(1, 1, 2, 3, 4, 5, 7, 9, 11, 7, 4, 6)	(1, 0, 1, 1, 1, 1, 2, 2, 2, 2, 3, 2)	−2
(4, 6)	[0, 0, 0, 0, 0, 0, 0, 2, 1]	(1, 2, 3, 4, 5, 6, 7, 8, 9, 6, 4, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 4, 2)	2
(4, 6)	[0, 0, 0, 0, 0, 0, 1, 0, 2]	(1, 2, 3, 4, 5, 6, 7, 8, 10, 6, 4, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 4, 2)	0
(4, 6)	[1, 0, 0, 0, 0, 1, 0, 0, 2]	(1, 1, 2, 3, 4, 5, 6, 8, 10, 6, 4, 6)	(1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 4, 2)	2
(5, 6)	[0, 0, 0, 1, 0, 0, 0, 0, 0]	(1, 2, 3, 4, 5, 7, 9, 11, 13, 9, 5, 6)	(1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 1)	−6
(5, 6)	[0, 2, 0, 0, 0, 0, 0, 0, 0]	(1, 1, 1, 3, 5, 7, 9, 11, 13, 9, 5, 6)	(1, 0, 0, 2, 2, 2, 2, 2, 2, 2, 2, 1)	−2
(5, 6)	[1, 0, 1, 0, 0, 0, 0, 0, 0]	(1, 1, 2, 3, 5, 7, 9, 11, 13, 9, 5, 6)	(1, 0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 1)	−4
(5, 6)	[0, 0, 0, 0, 0, 0, 2, 0, 0]	(1, 2, 3, 4, 5, 6, 7, 8, 11, 8, 5, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 1)	0
(5, 6)	[0, 0, 0, 0, 0, 1, 0, 1, 0]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 8, 5, 6)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, 1)	−2
(5, 6)	[0, 0, 0, 0, 1, 0, 0, 0, 1]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 8, 5, 6)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 1)	−4
(5, 6)	[0, 1, 0, 1, 0, 0, 0, 1, 0]	(1, 1, 1, 2, 3, 5, 7, 9, 11, 8, 5, 6)	(1, 0, 0, 1, 1, 2, 2, 2, 2, 3, 3, 1)	2
(5, 6)	[0, 1, 1, 0, 0, 0, 0, 0, 1]	(1, 1, 1, 2, 4, 6, 8, 10, 12, 8, 5, 6)	(1, 0, 0, 1, 2, 2, 2, 2, 2, 2, 3, 1)	0
(5, 6)	[1, 0, 0, 0, 0, 1, 1, 0, 0]	(1, 1, 2, 3, 4, 5, 6, 8, 11, 8, 5, 6)	(1, 0, 1, 1, 1, 1, 1, 2, 3, 3, 3, 1)	2
(5, 6)	[1, 0, 0, 0, 1, 0, 0, 1, 0]	(1, 1, 2, 3, 4, 5, 7, 9, 11, 8, 5, 6)	(1, 0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 1)	0
(5, 6)	[1, 0, 0, 1, 0, 0, 0, 0, 1]	(1, 1, 2, 3, 4, 6, 8, 10, 12, 8, 5, 6)	(1, 0, 1, 1, 1, 2, 2, 2, 2, 2, 3, 1)	−2
(5, 6)	[0, 0, 0, 0, 0, 0, 1, 1, 1]	(1, 2, 3, 4, 5, 6, 7, 8, 10, 7, 5, 6)	(1, 1, 1, 1, 1, 1, 1, 1, 2, 3, 4, 1)	2
(5, 6)	[0, 0, 0, 0, 0, 1, 0, 0, 2]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 7, 5, 6)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 4, 1)	0
(5, 6)	[1, 0, 0, 0, 1, 0, 0, 0, 2]	(1, 1, 2, 3, 4, 5, 7, 9, 11, 7, 5, 6)	(1, 0, 1, 1, 1, 1, 2, 2, 2, 2, 4, 1)	2
(6, 6)	[0, 0, 1, 0, 0, 0, 0, 0, 0]	(1, 2, 3, 4, 6, 8, 10, 12, 14, 10, 6, 6)	(1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 0)	−4
(6, 6)	[1, 1, 0, 0, 0, 0, 0, 0, 0]	(1, 1, 2, 4, 6, 8, 10, 12, 14, 10, 6, 6)	(1, 0, 1, 2, 2, 2, 2, 2, 2, 2, 2, 0)	−2
(6, 6)	[0, 0, 0, 0, 0, 1, 1, 0, 0]	(1, 2, 3, 4, 5, 6, 7, 9, 12, 9, 6, 6)	(1, 1, 1, 1, 1, 1, 1, 2, 3, 3, 3, 0)	2
(6, 6)	[0, 0, 0, 0, 1, 0, 0, 1, 0]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 9, 6, 6)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 3, 0)	0
(6, 6)	[0, 0, 0, 1, 0, 0, 0, 0, 1]	(1, 2, 3, 4, 5, 7, 9, 11, 13, 9, 6, 6)	(1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 0)	−2
(6, 6)	[0, 2, 0, 0, 0, 0, 0, 0, 1]	(1, 1, 1, 3, 5, 7, 9, 11, 13, 9, 6, 6)	(1, 0, 0, 2, 2, 2, 2, 2, 2, 2, 3, 0)	2
(6, 6)	[1, 0, 0, 1, 0, 0, 0, 1, 0]	(1, 1, 2, 3, 4, 6, 8, 10, 12, 9, 6, 6)	(1, 0, 1, 1, 1, 2, 2, 2, 2, 3, 3, 0)	2
(6, 6)	[1, 0, 1, 0, 0, 0, 0, 0, 1]	(1, 1, 2, 3, 5, 7, 9, 11, 13, 9, 6, 6)	(1, 0, 1, 1, 2, 2, 2, 2, 2, 2, 3, 0)	0
(6, 6)	[0, 0, 0, 0, 1, 0, 0, 0, 2]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 8, 6, 6)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 4, 0)	2
(2, 7)	[0, 0, 0, 0, 1, 0, 0, 0, 0]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 7, 2, 7)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 5)	2
(3, 7)	[0, 0, 0, 1, 0, 0, 0, 0, 0]	(1, 2, 3, 4, 5, 7, 9, 11, 13, 8, 3, 7)	(1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 4)	−4
(3, 7)	[0, 2, 0, 0, 0, 0, 0, 0, 0]	(1, 1, 1, 3, 5, 7, 9, 11, 13, 8, 3, 7)	(1, 0, 0, 2, 2, 2, 2, 2, 2, 2, 2, 4)	0
(3, 7)	[1, 0, 1, 0, 0, 0, 0, 0, 0]	(1, 1, 2, 3, 5, 7, 9, 11, 13, 8, 3, 7)	(1, 0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 4)	−2
(3, 7)	[0, 0, 0, 0, 0, 0, 2, 0, 0]	(1, 2, 3, 4, 5, 6, 7, 8, 11, 7, 3, 7)	(1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 4)	2
(3, 7)	[0, 0, 0, 0, 0, 1, 0, 1, 0]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 7, 3, 7)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, 4)	0
(3, 7)	[0, 0, 0, 0, 1, 0, 0, 0, 1]	(1, 2, 3, 4, 5, 6, 8, 10, 12, 7, 3, 7)	(1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 4)	−2
(3, 7)	[0, 1, 1, 0, 0, 0, 0, 0, 1]	(1, 1, 1, 2, 4, 6, 8, 10, 12, 7, 3, 7)	(1, 0, 0, 1, 2, 2, 2, 2, 2, 2, 3, 4)	2
(3, 7)	[1, 0, 0, 0, 1, 0, 0, 1, 0]	(1, 1, 2, 3, 4, 5, 7, 9, 11, 7, 3, 7)	(1, 0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4)	2
(3, 7)	[1, 0, 0, 1, 0, 0, 0, 0, 1]	(1, 1, 2, 3, 4, 6, 8, 10, 12, 7, 3, 7)	(1, 0, 1, 1, 1, 2, 2, 2, 2, 2, 3, 4)	0
(3, 7)	[0, 0, 0, 0, 0, 1, 0, 0, 2]	(1, 2, 3, 4, 5, 6, 7, 9, 11, 6, 3, 7)	(1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 4, 4)	2

Table 26 — Continued on next page

Table 26 — Continued from previous page

Level (m_{10}, m_{11})	A_9 weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
$(4,7)$	$[0,0,1,0,0,0,0,0,0]$	$(1,2,3,4,6,8,10,12,14,9,4,7)$	$(1,1,1,1,2,2,2,2,2,2,2,3)$	-8
$(4,7)$	$[1,1,0,0,0,0,0,0,0]$	$(1,1,2,4,6,8,10,12,14,9,4,7)$	$(1,0,1,2,2,2,2,2,2,2,2,3)$	-6
$(4,7)$	$[0,0,0,0,0,1,1,0,0]$	$(1,2,3,4,5,6,7,9,12,8,4,7)$	$(1,1,1,1,1,1,1,2,3,3,3,3)$	-2
$(4,7)$	$[0,0,0,0,1,0,0,1,0]$	$(1,2,3,4,5,6,8,10,12,8,4,7)$	$(1,1,1,1,1,1,2,2,2,3,3,3)$	-4
$(4,7)$	$[0,0,0,1,0,0,0,0,1]$	$(1,2,3,4,5,7,9,11,13,8,4,7)$	$(1,1,1,1,1,2,2,2,2,2,3,3)$	-6
$(4,7)$	$[0,1,0,1,0,0,1,0,0]$	$(1,1,1,2,3,5,7,9,12,8,4,7)$	$(1,0,0,1,1,2,2,2,3,3,3,3)$	2
$(4,7)$	$[0,1,1,0,0,0,0,1,0]$	$(1,1,1,2,4,6,8,10,12,8,4,7)$	$(1,0,0,1,2,2,2,2,2,3,3,3)$	0
$(4,7)$	$[0,2,0,0,0,0,0,0,1]$	$(1,1,1,3,5,7,9,11,13,8,4,7)$	$(1,0,0,2,2,2,2,2,2,2,3,3)$	-2
$(4,7)$	$[1,0,0,0,0,2,0,0,0]$	$(1,1,2,3,4,5,6,9,12,8,4,7)$	$(1,0,1,1,1,1,1,3,3,3,3,3)$	2
$(4,7)$	$[1,0,0,0,1,0,1,0,0]$	$(1,1,2,3,4,5,7,9,12,8,4,7)$	$(1,0,1,1,1,1,2,2,3,3,3,3)$	0
$(4,7)$	$[1,0,0,1,0,0,0,1,0]$	$(1,1,2,3,4,6,8,10,12,8,4,7)$	$(1,0,1,1,1,2,2,2,2,3,3,3)$	-2
$(4,7)$	$[1,0,1,0,0,0,0,0,1]$	$(1,1,2,3,5,7,9,11,13,8,4,7)$	$(1,0,1,1,2,2,2,2,2,2,3,3)$	-4
$(4,7)$	$[0,0,0,0,0,0,2,0,1]$	$(1,2,3,4,5,6,7,8,11,7,4,7)$	$(1,1,1,1,1,1,1,1,3,3,4,3)$	2
$(4,7)$	$[0,0,0,0,0,1,0,1,1]$	$(1,2,3,4,5,6,7,9,11,7,4,7)$	$(1,1,1,1,1,1,1,2,2,3,4,3)$	0
$(4,7)$	$[0,0,0,0,1,0,0,0,2]$	$(1,2,3,4,5,6,8,10,12,7,4,7)$	$(1,1,1,1,1,1,2,2,2,2,4,3)$	-2
$(4,7)$	$[0,1,1,0,0,0,0,0,2]$	$(1,1,1,2,4,6,8,10,12,7,4,7)$	$(1,0,0,1,2,2,2,2,2,2,4,3)$	2
$(4,7)$	$[1,0,0,0,1,0,0,1,1]$	$(1,1,2,3,4,5,7,9,11,7,4,7)$	$(1,0,1,1,1,1,2,2,2,3,4,3)$	2
$(4,7)$	$[1,0,0,1,0,0,0,0,2]$	$(1,1,2,3,4,6,8,10,12,7,4,7)$	$(1,0,1,1,1,2,2,2,2,2,4,3)$	0

Table 26: Low level weights in the *IIA supergravity* of the l_1 representation of E_{11}.

Level (m_9, m_{10})	A_9 weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
$(0,0)$	$[1,0,0,0,0,0,0,0,0]$	$(1,0,0,0,0,0,0,0,0,0,0,0)$	$(1,-1,0,0,0,0,0,0,0,0,0,0)$	2
$(1,0)$	$[0,0,0,0,0,0,0,0,1]$	$(1,1,1,1,1,1,1,1,1,1,0,0)$	$(1,0,0,0,0,0,0,0,0,0,-1,0)$	2
$(1,1)$	$[0,0,0,0,0,0,0,0,1]$	$(1,1,1,1,1,1,1,1,1,1,1,0)$	$(1,0,0,0,0,0,0,0,0,0,0,-1)$	2
$(2,1)$	$[0,0,0,0,0,0,1,0,0]$	$(1,1,1,1,1,1,1,1,2,2,1,1)$	$(1,0,0,0,0,0,0,0,1,1,0,0)$	2
$(3,1)$	$[0,0,0,0,1,0,0,0,0]$	$(1,1,1,1,1,1,2,3,4,3,1,2)$	$(1,0,0,0,0,0,1,1,1,1,0,1)$	2
$(3,2)$	$[0,0,0,0,1,0,0,0,0]$	$(1,1,1,1,1,1,2,3,4,3,2,2)$	$(1,0,0,0,0,0,1,1,1,1,1,0)$	2
$(4,1)$	$[0,0,1,0,0,0,0,0,0]$	$(1,1,1,1,2,3,4,5,6,4,1,3)$	$(1,0,0,0,1,1,1,1,1,1,0,2)$	2
$(4,2)$	$[0,0,1,0,0,0,0,0,0]$	$(1,1,1,1,2,3,4,5,6,4,2,3)$	$(1,0,0,0,1,1,1,1,1,1,1,1)$	0
$(4,2)$	$[0,0,0,1,0,0,0,0,1]$	$(1,1,1,1,1,2,3,4,5,4,2,2)$	$(1,0,0,0,0,1,1,1,1,1,0,0)$	2
$(4,3)$	$[0,0,1,0,0,0,0,0,0]$	$(1,1,1,1,2,3,4,5,6,4,3,3)$	$(1,0,0,0,1,1,1,1,1,1,2,0)$	2
$(5,1)$	$[1,0,0,0,0,0,0,0,0]$	$(1,1,2,3,4,5,6,7,8,5,1,4)$	$(1,0,1,1,1,1,1,1,1,1,0,3)$	2
$(5,2)$	$[1,0,0,0,0,0,0,0,0]$	$(1,1,2,3,4,5,6,7,8,5,2,4)$	$(1,0,1,1,1,1,1,1,1,1,1,2)$	-2
$(5,2)$	$[0,0,1,0,0,0,0,1,0]$	$(1,1,1,1,2,3,4,5,6,5,2,3)$	$(1,0,0,0,1,1,1,1,1,2,0,1)$	2
$(5,2)$	$[0,1,0,0,0,0,0,0,1]$	$(1,1,1,2,3,4,5,6,7,5,2,3)$	$(1,0,0,1,1,1,1,1,1,1,0,1)$	0
$(5,3)$	$[1,0,0,0,0,0,0,0,0]$	$(1,1,2,3,4,5,6,7,8,5,3,4)$	$(1,0,1,1,1,1,1,1,1,1,2,1)$	-2
$(5,3)$	$[0,0,1,0,0,0,0,1,0]$	$(1,1,1,1,2,3,4,5,6,5,3,3)$	$(1,0,0,0,1,1,1,1,1,2,1,0)$	2
$(5,3)$	$[0,1,0,0,0,0,0,0,1]$	$(1,1,1,2,3,4,5,6,7,5,3,3)$	$(1,0,0,1,1,1,1,1,1,1,1,0)$	0
$(5,4)$	$[1,0,0,0,0,0,0,0,0]$	$(1,1,2,3,4,5,6,7,8,5,4,4)$	$(1,0,1,1,1,1,1,1,1,1,3,0)$	2
$(6,2)$	$[0,0,0,0,0,0,0,0,1]$	$(1,2,3,4,5,6,7,8,9,6,2,4)$	$(1,1,1,1,1,1,1,1,1,1,0,2)$	-2
$(6,2)$	$[0,1,0,0,0,0,1,0,0]$	$(1,1,1,2,3,4,5,6,8,6,2,4)$	$(1,0,0,1,1,1,1,1,2,2,0,2)$	2
$(6,2)$	$[1,0,0,0,0,0,0,1,0]$	$(1,1,2,3,4,5,6,7,8,6,2,4)$	$(1,0,1,1,1,1,1,1,1,2,0,2)$	0
$(6,2)$	$[1,0,0,0,0,0,0,0,2]$	$(1,1,2,3,4,5,6,7,8,6,2,3)$	$(1,0,1,1,1,1,1,1,1,-1,1)$	2
$(6,3)$	$[0,0,0,0,0,0,0,0,1]$	$(1,2,3,4,5,6,7,8,9,6,3,4)$	$(1,1,1,1,1,1,1,1,1,1,1,1)$	-4
$(6,3)$	$[0,0,1,0,0,1,0,0,0]$	$(1,1,1,1,2,3,4,6,8,6,3,4)$	$(1,0,0,0,1,1,1,2,2,2,1,1)$	2
$(6,3)$	$[0,1,0,0,0,0,1,0,0]$	$(1,1,1,2,3,4,5,6,8,6,3,4)$	$(1,0,0,1,1,1,1,1,2,2,1,1)$	0
$(6,3)$	$[1,0,0,0,0,0,0,1,0]$	$(1,1,2,3,4,5,6,7,8,6,3,4)$	$(1,0,1,1,1,1,1,1,1,2,1,1)$	-2
$(6,3)$	$[0,1,0,0,0,0,0,1,1]$	$(1,1,1,2,3,4,5,6,7,6,3,3)$	$(1,0,0,1,1,1,1,1,1,2,0,0)$	2
$(6,3)$	$[1,0,0,0,0,0,0,0,2]$	$(1,1,2,3,4,5,6,7,8,6,3,3)$	$(1,0,1,1,1,1,1,1,1,1,0,0)$	0
$(6,4)$	$[0,0,0,0,0,0,0,0,1]$	$(1,2,3,4,5,6,7,8,9,6,4,4)$	$(1,1,1,1,1,1,1,1,1,1,2,0)$	-2
$(6,4)$	$[0,1,0,0,0,0,1,0,0]$	$(1,1,1,2,3,4,5,6,8,6,4,4)$	$(1,0,0,1,1,1,1,1,2,2,2,0)$	2
$(6,4)$	$[1,0,0,0,0,0,0,1,0]$	$(1,1,2,3,4,5,6,7,8,6,4,4)$	$(1,0,1,1,1,1,1,1,1,2,2,0)$	0

Table 27 — Continued on next page

Table 27 — Continued from previous page

Level (m_9, m_{10})	A_9 weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
$(6,4)$	$[1,0,0,0,0,0,0,0,2]$	$(1,1,2,3,4,5,6,7,8,6,4,3)$	$(1,0,1,1,1,1,1,1,1,1,1,-1)$	2
$(7,2)$	$[0,0,0,0,0,0,1,0,0]$	$(1,2,3,4,5,6,7,8,10,7,2,5)$	$(1,1,1,1,1,1,1,1,2,2,0,3)$	0
$(7,2)$	$[1,0,0,0,0,1,0,0,0]$	$(1,1,2,3,4,5,6,8,10,7,2,5)$	$(1,0,1,1,1,1,1,2,2,2,0,3)$	2
$(7,2)$	$[0,0,0,0,0,0,0,1,1]$	$(1,2,3,4,5,6,7,8,9,7,2,4)$	$(1,1,1,1,1,1,1,1,1,2,-1,2)$	2
$(7,3)$	$[0,0,0,0,0,0,1,0,0]$	$(1,2,3,4,5,6,7,8,10,7,3,5)$	$(1,1,1,1,1,1,1,1,2,2,1,2)$	-4
$(7,3)$	$[0,0,1,1,0,0,0,0,0]$	$(1,1,1,1,2,4,6,8,10,7,3,5)$	$(1,0,0,0,1,2,2,2,2,2,1,2)$	2
$(7,3)$	$[0,1,0,0,1,0,0,0,0]$	$(1,1,1,2,3,4,6,8,10,7,3,5)$	$(1,0,0,1,1,1,2,2,2,2,1,2)$	0
$(7,3)$	$[1,0,0,0,0,1,0,0,0]$	$(1,1,2,3,4,5,6,8,10,7,3,5)$	$(1,0,1,1,1,1,1,2,2,2,1,2)$	-2
$(7,3)$	$[0,0,0,0,0,0,0,1,1]$	$(1,2,3,4,5,6,7,8,9,7,3,4)$	$(1,1,1,1,1,1,1,1,1,2,0,1)$	-2
$(7,3)$	$[0,1,0,0,0,1,0,0,1]$	$(1,1,1,2,3,4,5,7,9,7,3,4)$	$(1,0,0,1,1,1,1,2,2,2,0,1)$	2
$(7,3)$	$[1,0,0,0,0,0,0,2,0]$	$(1,1,2,3,4,5,6,7,8,7,3,4)$	$(1,0,1,1,1,1,1,1,1,3,0,1)$	2
$(7,3)$	$[1,0,0,0,0,0,1,0,1]$	$(1,1,2,3,4,5,6,7,9,7,3,4)$	$(1,0,1,1,1,1,1,1,2,2,0,1)$	0
$(7,3)$	$[0,0,0,0,0,0,0,0,3]$	$(1,2,3,4,5,6,7,8,9,7,3,3)$	$(1,1,1,1,1,1,1,1,1,1,-1,0)$	2
$(7,4)$	$[0,0,0,0,0,0,1,0,0]$	$(1,2,3,4,5,6,7,8,10,7,4,5)$	$(1,1,1,1,1,1,1,1,2,2,2,1)$	-4
$(7,4)$	$[0,0,1,1,0,0,0,0,0]$	$(1,1,1,1,2,4,6,8,10,7,4,5)$	$(1,0,0,0,1,2,2,2,2,2,2,1)$	2
$(7,4)$	$[0,1,0,0,1,0,0,0,0]$	$(1,1,1,2,3,4,6,8,10,7,4,5)$	$(1,0,0,1,1,1,2,2,2,2,2,1)$	0
$(7,4)$	$[1,0,0,0,0,1,0,0,0]$	$(1,1,2,3,4,5,6,8,10,7,4,5)$	$(1,0,1,1,1,1,1,2,2,2,2,1)$	-2
$(7,4)$	$[0,0,0,0,0,0,0,1,1]$	$(1,2,3,4,5,6,7,8,9,7,4,4)$	$(1,1,1,1,1,1,1,1,1,2,1,0)$	-2
$(7,4)$	$[0,1,0,0,0,1,0,0,1]$	$(1,1,1,2,3,4,5,7,9,7,4,4)$	$(1,0,0,1,1,1,1,2,2,2,1,0)$	2
$(7,4)$	$[1,0,0,0,0,0,0,2,0]$	$(1,1,2,3,4,5,6,7,8,7,4,4)$	$(1,0,1,1,1,1,1,1,1,3,1,0)$	2
$(7,4)$	$[1,0,0,0,0,0,1,0,1]$	$(1,1,2,3,4,5,6,7,9,7,4,4)$	$(1,0,1,1,1,1,1,1,2,2,1,0)$	0
$(7,4)$	$[0,0,0,0,0,0,0,0,3]$	$(1,2,3,4,5,6,7,8,9,7,4,3)$	$(1,1,1,1,1,1,1,1,1,1,0,-1)$	2
$(7,5)$	$[0,0,0,0,0,0,1,0,0]$	$(1,2,3,4,5,6,7,8,10,7,5,5)$	$(1,1,1,1,1,1,1,1,2,2,3,0)$	0
$(7,5)$	$[1,0,0,0,0,1,0,0,0]$	$(1,1,2,3,4,5,6,8,10,7,5,5)$	$(1,0,1,1,1,1,1,2,2,2,3,0)$	2
$(7,5)$	$[0,0,0,0,0,0,0,1,1]$	$(1,2,3,4,5,6,7,8,9,7,5,4)$	$(1,1,1,1,1,1,1,1,1,2,2,-1)$	2
$(8,2)$	$[0,0,0,0,1,0,0,0,0]$	$(1,2,3,4,5,6,8,10,12,8,2,6)$	$(1,1,1,1,1,1,2,2,2,2,0,4)$	2
$(8,3)$	$[0,0,0,0,1,0,0,0,0]$	$(1,2,3,4,5,6,8,10,12,8,3,6)$	$(1,1,1,1,1,1,2,2,2,2,1,3)$	-4
$(8,3)$	$[0,1,1,0,0,0,0,0,0]$	$(1,1,1,2,4,6,8,10,12,8,3,6)$	$(1,0,0,1,2,2,2,2,2,2,1,3)$	0
$(8,3)$	$[1,0,0,1,0,0,0,0,0]$	$(1,1,2,3,4,6,8,10,12,8,3,6)$	$(1,0,1,1,1,2,2,2,2,2,1,3)$	-2
$(8,3)$	$[0,0,0,0,0,0,1,1,0]$	$(1,2,3,4,5,6,7,8,10,8,3,5)$	$(1,1,1,1,1,1,1,1,2,3,0,2)$	0
$(8,3)$	$[0,0,0,0,0,1,0,0,1]$	$(1,2,3,4,5,6,7,9,11,8,3,5)$	$(1,1,1,1,1,1,1,2,2,2,0,2)$	-2
$(8,3)$	$[0,1,0,1,0,0,0,0,1]$	$(1,1,1,2,3,5,7,9,11,8,3,5)$	$(1,0,0,1,1,2,2,2,2,2,0,2)$	2
$(8,3)$	$[1,0,0,0,0,1,0,1,0]$	$(1,1,2,3,4,5,6,8,10,8,3,5)$	$(1,0,1,1,1,1,1,2,2,3,0,2)$	2
$(8,3)$	$[1,0,0,0,1,0,0,0,1]$	$(1,1,2,3,4,5,7,9,11,8,3,5)$	$(1,0,1,1,1,1,2,2,2,2,0,2)$	0
$(8,3)$	$[0,0,0,0,0,0,1,0,2]$	$(1,2,3,4,5,6,7,8,10,8,3,4)$	$(1,1,1,1,1,1,1,1,2,2,-1,1)$	2
$(8,4)$	$[0,0,0,0,1,0,0,0,0]$	$(1,2,3,4,5,6,8,10,12,8,4,6)$	$(1,1,1,1,1,1,2,2,2,2,2,2)$	-6
$(8,4)$	$[0,1,1,0,0,0,0,0,0]$	$(1,1,1,2,4,6,8,10,12,8,4,6)$	$(1,0,0,1,2,2,2,2,2,2,2,2)$	-2
$(8,4)$	$[1,0,0,1,0,0,0,0,0]$	$(1,1,2,3,4,6,8,10,12,8,4,6)$	$(1,0,1,1,1,2,2,2,2,2,2,2)$	-4
$(8,4)$	$[0,0,0,0,0,0,1,1,0]$	$(1,2,3,4,5,6,7,8,10,8,4,5)$	$(1,1,1,1,1,1,1,1,2,3,1,1)$	-2
$(8,4)$	$[0,0,0,0,0,1,0,0,1]$	$(1,2,3,4,5,6,7,9,11,8,4,5)$	$(1,1,1,1,1,1,1,2,2,2,1,1)$	-4
$(8,4)$	$[0,0,2,0,0,0,0,0,1]$	$(1,1,1,1,3,5,7,9,11,8,4,5)$	$(1,0,0,0,2,2,2,2,2,2,1,1)$	2
$(8,4)$	$[0,1,0,0,1,0,0,1,0]$	$(1,1,1,2,3,4,6,8,10,8,4,5)$	$(1,0,0,1,1,1,2,2,2,3,1,1)$	2
$(8,4)$	$[0,1,0,1,0,0,0,0,1]$	$(1,1,1,2,3,5,7,9,11,8,4,5)$	$(1,0,0,1,1,2,2,2,2,2,1,1)$	0
$(8,4)$	$[1,0,0,0,0,0,2,0,0]$	$(1,1,2,3,4,5,6,7,10,8,4,5)$	$(1,0,1,1,1,1,1,1,3,3,1,1)$	2
$(8,4)$	$[1,0,0,0,0,1,0,1,0]$	$(1,1,2,3,4,5,6,8,10,8,4,5)$	$(1,0,1,1,1,1,1,2,2,3,1,1)$	0
$(8,4)$	$[1,0,0,0,1,0,0,0,1]$	$(1,1,2,3,4,5,7,9,11,8,4,5)$	$(1,0,1,1,1,1,2,2,2,2,1,1)$	-2
$(8,4)$	$[0,0,0,0,0,0,0,2,1]$	$(1,2,3,4,5,6,7,8,9,8,4,4)$	$(1,1,1,1,1,1,1,1,1,3,0,0)$	2
$(8,4)$	$[0,0,0,0,0,0,1,0,2]$	$(1,2,3,4,5,6,7,8,10,8,4,4)$	$(1,1,1,1,1,1,1,1,2,2,0,0)$	0
$(8,4)$	$[1,0,0,0,0,1,0,0,2]$	$(1,1,2,3,4,5,6,8,10,8,4,4)$	$(1,0,1,1,1,1,1,2,2,2,0,0)$	2
$(8,5)$	$[0,0,0,0,1,0,0,0,0]$	$(1,2,3,4,5,6,8,10,12,8,5,6)$	$(1,1,1,1,1,1,2,2,2,2,3,1)$	-4
$(8,5)$	$[0,1,1,0,0,0,0,0,0]$	$(1,1,1,2,4,6,8,10,12,8,5,6)$	$(1,0,0,1,2,2,2,2,2,2,3,1)$	0
$(8,5)$	$[1,0,0,1,0,0,0,0,0]$	$(1,1,2,3,4,6,8,10,12,8,5,6)$	$(1,0,1,1,1,2,2,2,2,2,3,1)$	-2
$(8,5)$	$[0,0,0,0,0,0,1,1,0]$	$(1,2,3,4,5,6,7,8,10,8,5,5)$	$(1,1,1,1,1,1,1,1,2,3,2,0)$	0
$(8,5)$	$[0,0,0,0,0,1,0,0,1]$	$(1,2,3,4,5,6,7,9,11,8,5,5)$	$(1,1,1,1,1,1,1,2,2,2,2,0)$	-2
$(8,5)$	$[0,1,0,1,0,0,0,0,1]$	$(1,1,1,2,3,5,7,9,11,8,5,5)$	$(1,0,0,1,1,2,2,2,2,2,2,0)$	2

Table 27 — Continued on next page

Level (m_9, m_{10})	A_9 weights	E_{12} Root (α_i basis)	E_{12} Root (e_i basis)	Root length squared
$(8,5)$	$[1,0,0,0,0,1,0,1,0]$	$(1,1,2,3,4,5,6,8,10,8,5,5)$	$(1,0,1,1,1,1,1,2,2,3,2,0)$	2
$(8,5)$	$[1,0,0,0,1,0,0,0,1]$	$(1,1,2,3,4,5,7,9,11,8,5,5)$	$(1,0,1,1,1,1,2,2,2,2,2,0)$	0
$(8,5)$	$[0,0,0,0,0,0,1,0,2]$	$(1,2,3,4,5,6,7,8,10,8,5,4)$	$(1,1,1,1,1,1,1,1,2,2,1,-1)$	2
$(8,6)$	$[0,0,0,0,1,0,0,0,0]$	$(1,2,3,4,5,6,8,10,12,8,6,6)$	$(1,1,1,1,1,1,2,2,2,2,4,0)$	2
$(9,3)$	$[0,0,1,0,0,0,0,0,0]$	$(1,2,3,4,6,8,10,12,14,9,3,7)$	$(1,1,1,1,2,2,2,2,2,2,1,4)$	-4
$(9,3)$	$[1,1,0,0,0,0,0,0,0]$	$(1,1,2,4,6,8,10,12,14,9,3,7)$	$(1,0,1,2,2,2,2,2,2,2,1,4)$	-2
$(9,3)$	$[0,0,0,0,0,1,1,0,0]$	$(1,2,3,4,5,6,7,9,12,9,3,6)$	$(1,1,1,1,1,1,1,2,3,3,0,3)$	2
$(9,3)$	$[0,0,0,0,1,0,0,1,0]$	$(1,2,3,4,5,6,8,10,12,9,3,6)$	$(1,1,1,1,1,1,2,2,2,3,0,3)$	0
$(9,3)$	$[0,0,0,1,0,0,0,0,1]$	$(1,2,3,4,5,7,9,11,13,9,3,6)$	$(1,1,1,1,1,2,2,2,2,2,0,3)$	-2
$(9,3)$	$[0,2,0,0,0,0,0,0,1]$	$(1,1,1,3,5,7,9,11,13,9,3,6)$	$(1,0,0,2,2,2,2,2,2,2,0,3)$	2
$(9,3)$	$[1,0,0,1,0,0,0,1,0]$	$(1,1,2,3,4,6,8,10,12,9,3,6)$	$(1,0,1,1,1,2,2,2,2,3,0,3)$	2
$(9,3)$	$[1,0,1,0,0,0,0,0,1]$	$(1,1,2,3,5,7,9,11,13,9,3,6)$	$(1,0,1,1,2,2,2,2,2,2,0,3)$	0
$(9,3)$	$[0,0,0,0,1,0,0,0,2]$	$(1,2,3,4,5,6,8,10,12,9,3,5)$	$(1,1,1,1,1,1,2,2,2,2,-1,2)$	2
$(9,4)$	$[0,0,1,0,0,0,0,0,0]$	$(1,2,3,4,6,8,10,12,14,9,4,7)$	$(1,1,1,1,2,2,2,2,2,2,2,3)$	-8
$(9,4)$	$[1,1,0,0,0,0,0,0,0]$	$(1,1,2,4,6,8,10,12,14,9,4,7)$	$(1,0,1,2,2,2,2,2,2,2,2,3)$	-6
$(9,4)$	$[0,0,0,0,0,1,1,0,0]$	$(1,2,3,4,5,6,7,9,12,9,4,6)$	$(1,1,1,1,1,1,1,2,3,3,1,2)$	-2
$(9,4)$	$[0,0,0,0,1,0,0,1,0]$	$(1,2,3,4,5,6,8,10,12,9,4,6)$	$(1,1,1,1,1,1,2,2,2,3,1,2)$	-4
$(9,4)$	$[0,0,0,1,0,0,0,0,1]$	$(1,2,3,4,5,7,9,11,13,9,4,6)$	$(1,1,1,1,1,2,2,2,2,2,1,2)$	-6
$(9,4)$	$[0,1,0,1,0,0,1,0,0]$	$(1,1,1,2,3,5,7,9,12,9,4,6)$	$(1,0,0,1,1,2,2,2,3,3,1,2)$	2
$(9,4)$	$[0,1,1,0,0,0,0,1,0]$	$(1,1,1,2,4,6,8,10,12,9,4,6)$	$(1,0,0,1,2,2,2,2,2,3,1,2)$	0
$(9,4)$	$[0,2,0,0,0,0,0,0,1]$	$(1,1,1,3,5,7,9,11,13,9,4,6)$	$(1,0,0,2,2,2,2,2,2,2,1,2)$	-2
$(9,4)$	$[1,0,0,0,0,2,0,0,0]$	$(1,1,2,3,4,5,6,9,12,9,4,6)$	$(1,0,1,1,1,1,1,3,3,3,1,2)$	2
$(9,4)$	$[1,0,0,0,1,0,1,0,0]$	$(1,1,2,3,4,5,7,9,12,9,4,6)$	$(1,0,1,1,1,1,2,2,3,3,1,2)$	0
$(9,4)$	$[1,0,0,1,0,0,0,1,0]$	$(1,1,2,3,4,6,8,10,12,9,4,6)$	$(1,0,1,1,1,2,2,2,2,3,1,2)$	-2
$(9,4)$	$[1,0,1,0,0,0,0,0,1]$	$(1,1,2,3,5,7,9,11,13,9,4,6)$	$(1,0,1,1,2,2,2,2,2,2,1,2)$	-4
$(9,4)$	$[0,0,0,0,0,0,2,0,1]$	$(1,2,3,4,5,6,7,8,11,9,4,5)$	$(1,1,1,1,1,1,1,1,3,3,0,1)$	2
$(9,4)$	$[0,0,0,0,0,1,0,1,1]$	$(1,2,3,4,5,6,7,9,11,9,4,5)$	$(1,1,1,1,1,1,1,2,2,3,0,1)$	0
$(9,4)$	$[0,0,0,0,1,0,0,0,2]$	$(1,2,3,4,5,6,8,10,12,9,4,5)$	$(1,1,1,1,1,1,2,2,2,2,0,1)$	-2
$(9,4)$	$[0,1,1,0,0,0,0,0,2]$	$(1,1,1,2,4,6,8,10,12,9,4,5)$	$(1,0,0,1,2,2,2,2,2,2,0,1)$	2
$(9,4)$	$[1,0,0,0,1,0,0,1,1]$	$(1,1,2,3,4,5,7,9,11,9,4,5)$	$(1,0,1,1,1,1,2,2,2,3,0,1)$	2
$(9,4)$	$[1,0,0,1,0,0,0,0,2]$	$(1,1,2,3,4,6,8,10,12,9,4,5)$	$(1,0,1,1,1,2,2,2,2,2,0,1)$	0
$(9,5)$	$[0,0,1,0,0,0,0,0,0]$	$(1,2,3,4,6,8,10,12,14,9,5,7)$	$(1,1,1,1,2,2,2,2,2,2,3,2)$	-8
$(9,5)$	$[1,1,0,0,0,0,0,0,0]$	$(1,1,2,4,6,8,10,12,14,9,5,7)$	$(1,0,1,2,2,2,2,2,2,2,3,2)$	-6
$(9,5)$	$[0,0,0,0,0,1,1,0,0]$	$(1,2,3,4,5,6,7,9,12,9,5,6)$	$(1,1,1,1,1,1,1,2,3,3,2,1)$	-2
$(9,5)$	$[0,0,0,0,1,0,0,1,0]$	$(1,2,3,4,5,6,8,10,12,9,5,6)$	$(1,1,1,1,1,1,2,2,2,3,2,1)$	-4
$(9,5)$	$[0,0,0,1,0,0,0,0,1]$	$(1,2,3,4,5,7,9,11,13,9,5,6)$	$(1,1,1,1,1,2,2,2,2,2,2,1)$	-6
$(9,5)$	$[0,1,0,1,0,0,1,0,0]$	$(1,1,1,2,3,5,7,9,12,9,5,6)$	$(1,0,0,1,1,2,2,2,3,3,2,1)$	2
$(9,5)$	$[0,1,1,0,0,0,0,1,0]$	$(1,1,1,2,4,6,8,10,12,9,5,6)$	$(1,0,0,1,2,2,2,2,2,3,2,1)$	0
$(9,5)$	$[0,2,0,0,0,0,0,0,1]$	$(1,1,1,3,5,7,9,11,13,9,5,6)$	$(1,0,0,2,2,2,2,2,2,2,2,1)$	-2
$(9,5)$	$[1,0,0,0,0,2,0,0,0]$	$(1,1,2,3,4,5,6,9,12,9,5,6)$	$(1,0,1,1,1,1,1,3,3,3,2,1)$	2
$(9,5)$	$[1,0,0,0,1,0,1,0,0]$	$(1,1,2,3,4,5,7,9,12,9,5,6)$	$(1,0,1,1,1,1,2,2,3,3,2,1)$	0
$(9,5)$	$[1,0,0,1,0,0,0,1,0]$	$(1,1,2,3,4,6,8,10,12,9,5,6)$	$(1,0,1,1,1,2,2,2,2,3,2,1)$	-2
$(9,5)$	$[1,0,1,0,0,0,0,0,1]$	$(1,1,2,3,5,7,9,11,13,9,5,6)$	$(1,0,1,1,2,2,2,2,2,2,2,1)$	-4
$(9,5)$	$[0,0,0,0,0,0,2,0,1]$	$(1,2,3,4,5,6,7,8,11,9,5,5)$	$(1,1,1,1,1,1,1,1,3,3,1,0)$	2
$(9,5)$	$[0,0,0,0,0,1,0,1,1]$	$(1,2,3,4,5,6,7,9,11,9,5,5)$	$(1,1,1,1,1,1,1,2,2,3,1,0)$	0
$(9,5)$	$[0,0,0,0,1,0,0,0,2]$	$(1,2,3,4,5,6,8,10,12,9,5,5)$	$(1,1,1,1,1,1,2,2,2,2,1,0)$	-2
$(9,5)$	$[0,1,1,0,0,0,0,0,2]$	$(1,1,1,2,4,6,8,10,12,9,5,5)$	$(1,0,0,1,2,2,2,2,2,2,1,0)$	2
$(9,5)$	$[1,0,0,0,1,0,0,1,1]$	$(1,1,2,3,4,5,7,9,11,9,5,5)$	$(1,0,1,1,1,1,2,2,2,3,1,0)$	2
$(9,5)$	$[1,0,0,1,0,0,0,0,2]$	$(1,1,2,3,4,6,8,10,12,9,5,5)$	$(1,0,1,1,1,2,2,2,2,2,1,0)$	0

Table 27: Low level weights in the *IIB supergravity* decomposition of the l_1 representation of E_{11}.

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